On the projection methods for convex feasibility problems

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ABSTRACT. In this paper we consider a projection method for convex feasibility problem that is known to converge only weakly. Exploiting a property concerning the intersection of a family of convex closed sets, we present a condition that makes them strongly convergent, without additional assumptions.
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1 Introduction

Let H be a real Hilbert space with scalar product $\langle \cdot | \cdot \rangle$, norm $\|\cdot\|$, and distance d. In [Bre65] Bergman considered the classical projection method for finding a common point of m intersecting closed convex sets (M_i) in H. He showed that, given an arbitrary starting point $x_0 \in H$, the sequence generated by the projection algorithm converges weakly to a point in $M = \bigcap_{i=1}^{m} M_i$. Let us note that the results from [Bre65] are a generalization of a relaxation algorithm for solving a system of linear inequalities considered in [Agm54], [MS54]. Also, extensions of this relaxation algorithm has been given in [Ere69], [Jak66]. In [GPR67] certain regularity conditions on the sets were described that guaranteed strong convergence of the iterations. In recent papers, other conditions for strong convergence have been given, for example in [BK02], [BC01].

In this paper we present a simple condition that insure the strong convergence of the sequence generated by the projection algorithm.
Let $T : D \subset H \to H$ be a nonlinear mapping, and let *F(T)* denotes the set of fixed points of *T* in *D*. In the following we will assume that F(T)≠ Ø. According to [PW73], the mapping *T* is said to be quasi-nonexpansive if $\|Tx - x^*\| \leq \|x - x^*\|, \quad \forall x \in D, x^* \in F(T)$.

Remark 1. The notion of quasi-nonexpansivity has been introduced by Tricomi [Tri16] for real-valued fuctions and subsequently studied in [Tri16],[DM69] for mapping in Hilbert or Banach spaces.

Let *d(x,E)* denotes the distance between a point $x \in H$ and a set $E \subset H$, that is $d(x,E) = \inf_{x \in E} \|x - y\|$.

We shall use the following general theorem concerning the convergence of the simple iterates for quasi-nonexpansive mappings.

Theorem 1 *Suppose that* $T : D \subset H \to H$ *is a quasi-nonexpansive mapping and that F(T) is nonempty and closed. Let* $x_0 \in D$ *such that* $x_k = T^k_{x_0} \in D, k = 1,2,\cdots$. *Then the sequence* $\{x_k\}$ *converges to a fixed point of T if and only if there exists a subsequence* $\{x_{k_j}\}$ *of* $\{x_k\}$ *such that* $d(x_{k_j},F(T)) \to D$ *as* $j \to \infty$. ▪

Here, as usual, T^k denotes the *k* iterate of *T*.

Remark 2. Theorem 1 is a slight generalization of the first result of [PW73] and its proof is similar. Essentially, Theorem \ref{prima_teorema} replaced the condition of continuity of *T*, from the original result, by the condition of
closedness of *F(T)*. It is easy to see that the latter condition is weaker, and, as it will result, is essential for our development.

2 The main result

We first prove the following lemma.

Lemma 1. *Let* $M_i \subset H$ $(i = 1,\ldots,m)$ *be a family of convex sets such that* $Int \cap M_i$ *is nonempty and bounded and let* $\{x_k\}$ *be a sequence of* H *such that* $d(x_k, M_i) \to 0$ *as* $j \to \infty$ *for each i. Then* $d(x_k, \cap M_i) \to 0$, *as* $k \to \infty$

Proof. We assume that $o \in Int \cap M_i$. Then there exists a closed ball $D(o,r)=\{x \in H: \|x\| \le r\} \subset \cap M_i$. Let ε be a given real number, $0<\varepsilon<1$, and let C be a constant such that $\|x\| \le C-1$ for all $x \in \cap M_i$, which is possible, because $\cap M_i$ is bounded.

Since $d(x_k,M_i)\to 0$ as $k \to \infty$, for each index i, there exists a sequence $\{y_k^{(i)}\}_{k\in N} \subset M_i$ such that $\left\|y_k^{(i)} - x_k\right\| \to 0$ as $k \to \infty$. Let

$$z_k = (1-\frac{C}{\varepsilon})(y_k^{(i)} - x_k),\ k = 0,1,\cdots. \qquad (1)$$

There exists a number $k_i(\varepsilon)$ such that if $k \ge k_i(\varepsilon)$ then $\left\|y_k^{(i)}\right\| \le \frac{r}{\left|1-\frac{C}{\varepsilon}\right|}$ and so, that is $z_k \in \cap M_i$.

On the other hand, from Equation (1) we obtain

$$(1-\tfrac{\varepsilon}{C})x_k = \tfrac{\varepsilon}{C} z_k + (1-\tfrac{\varepsilon}{C})y_k^{(i)},$$

and for $k \ge k_i(\varepsilon)$ we have $(1-\frac{\varepsilon}{C})x_k \in M_i$, because $y_k^{(i)}, z_k \in M_i$ and M_i are convex.

Now, let $k_0(\varepsilon) = \max_i k_i(\varepsilon)$. Then, for $k \ge k_0(\varepsilon)$ it follows that $(1-\frac{\varepsilon}{C})x_k \in \cap M_i$ and

$$d(x_k, \cap M_i) \le \left\|x_k - (1-\tfrac{\varepsilon}{C})x_k\right\| = \tfrac{\varepsilon}{C-\varepsilon}\left\|(1-\tfrac{\varepsilon}{C})x_k\right\| < \varepsilon,$$

which end the proof. ▪

Apparently, the condition that $Int \cap M_i$ is nonempty and bounded is very strong. The following example shows that this condition cannot be replaced by the weaker condition $\bigcap M_i \neq \emptyset$, which seems to be more natural.

Example. Suppose that H is the real three-dimensional space, that the set M_i is a cone (A) and the set M_2 is a tangent plane ($ABCD$). The situation is depicted in Figure 1.

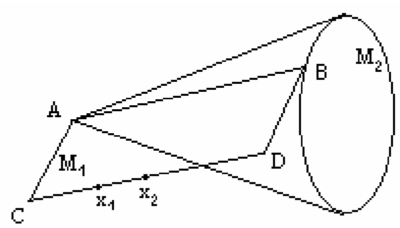

Figure 1: Example

The plane ($ABCD$) is tangent to the cone along the generatrix (AB) and hence $M_1 \cap M_2 = (AB)$. Now, let us consider a sequence $\{x_k\}$ in the plane (ABCD) such that $d(x_k,(AB)) = \delta = const.$ and $\|x_k\| \to \infty$ as $k \to \infty$. It is clear that $d(x_k, M_2) \to 0$ as $k \to \infty$ and $d(x_k,M_1) = 0$ for all k; but $d(x_k,M_1 \cap M_2) = \delta > 0$. Therefore, the conclusion of Lemma 2 is not true.
In the following, we shall suppose that M_i ($i=1,\dots, m$) are closed and convex sets of H. Let $P(x,i)$ be the projection of an $x \in H$ onto M_i and let i_x be the smallest index such that $\|x - P(x,i_x)\| = \max_i \|x - P(x,i)\|$. We define the mapping T: $H \to H$ by $T_x=P(x,i_x)$. It is easy to see that $x \in \cap M_i$ if and only if $Tx = x$; hence if and only if x is a fixed point of T. In other words, $F(T)=\cap M_i$.

Let $\lambda \in (0,2)$ and let $T_\lambda=I-\lambda(I-T)$, where I is the identity mapping of H into H. Obviously, $F(T_\lambda)=F(T)$.

Theorem 2. *Let M_i $(i=1,\dots, m)$ be a family of closed and convex sets of H such that Int $\cap M_i$ is nonempty and bounded. Then the sequence x_k given by $X_k = T_\lambda^k x_0$ converges (strongly) to a point of $\cap M_i$ for all $x_0 \in H$*

Proof. Since $F(T_\lambda)=\cap M_i$ is a closed set, it suffices to show that T_λ is quasi-nonexpansive on H and that $d(x_k,\cap M_i) \to 0$ as $k \to \infty$. Then Theorem 2 follows from Theorem 1.

Let $x \in H$ and $y \in \cap M_i$. Since $P(x,i_x)$ is the projection of x onto M_{i_x} and $y \in M_{i_x}$, we have

$$\langle Tx - y, x - Tx\rangle = \langle P(x,i_x) - x, x - P(x,i_x)\rangle \geq 0,$$

and

$$\begin{aligned}\|T_\lambda x - y\|^2 &= \|x-y\|^2 - 2\lambda\langle x-y, x-Tx\rangle + \lambda^2\|x-Tx\|^2 = \\ &= \|x-y\|^2 - \lambda(2-\lambda)\|x-Tx\|^2 - 2\lambda\langle Tx-y, x-Tx\rangle \quad (2)\\ &\leq \|x-y\|^2 - \lambda(2-\lambda)\|x-Tx\|^2.\end{aligned}$$

Therefore, we have

$$\|T_\lambda x - y\| \leq \|x-y\|, \forall x \in H, \ y \in \cap M_i,$$

and T_λ is quasi-nonexpansive on H.

Now, since $x_{k+1}= T_\lambda x_k$, from (3) it follows that the sequence $\{\|x_k - y\|\}$ is monotone decreasing and bounded, therefore $\|x_k - y\| \to \delta_y$ for each $y \in \cap M_i$. From Equation (2) we obtain

$$\|x_k - Tx_k\|^2 \le \frac{1}{\lambda(\lambda-2)}(\|x_k - y\|^2 - \|x_{k+1} - y\|^2)$$

and hence $\|x_k - Tx_k\| \to 0$ as $k \to \infty$. But $\|x - P(x,t)\| \le \|x - Tx\|$ for each i. Therefore $d(x_k, M_i)=\|x_k - P(x_k, i)\| \to 0$ as $k \to \infty$ and Theorem 2 is proved ▪

Remark 3. It is easy to see that the mapping T: $H \to H$ defined above $T_x=P(x,i_x)$ is not continuous. Indeed, let $m = 2$ and let x be a point of H such that $d(x,M_1)=d(x,M_2)$. Now, let $\{x_k\}$ be a sequence such that $x_k \to x$ as $k \to \infty$ and $d(x_k,M_1) < d(x_k,M_2)$ for all k. Then $\lim Tx_k = \lim P(x_k,M_2)= P(x,M_2)$; but $Tx = P(x,M_1)$, that is T is not continuous at x.

Theorem 2 extends to real Hilbert spaces a result of Eremin [EF69], which is in turn a generalization of the Motzkin-Agmon-Schoenberg relaxation algorithm for inequalities. Note that the conditions of Eremin's theorem, for the finite dimensional case are somewhat weaker; more precisely, it is required only that $\bigcap M_i \ne \emptyset$, while our theorem requires that $Int \bigcap M_i \ne \emptyset$ are bounded.

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