SOME PROPERTIES ON THE DETERMINISTIC FINITE AUTOMATA

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ABSTRACT: Let $A = (Q, \Sigma, \delta, q_0, F)$ be a deterministic finite automata, the purpose of this study is to determine some conditions on the transition function of A that ensure the existence of some properties. We give also a specific equivalence relation on set Q .
KEYWORDS: free monoid, deterministic finite automata, morphism of monoids, equivalence relation.

1. INTRODUCTION

Let Σ be a nonempty set of symbols called an alphabet. Any finite string over Σ is called a word over Σ.

The empty word, that is the word contains no letter, will be denoted by ε. Let Σ^* be the set of all words over Σ . If define the operation of two words x and y of Σ^* by juxtaposition, sometimes call it concatenation, then Σ^* is a semigroup with the identity ε. And hence Σ^* is a monoid. We call Σ^* the free monoid generated by Σ, see [Suy79]. A deterministic finite automata (DFA) is a quintuple $A = (Q, \Sigma, \delta, q_0, F)$, where

- Q is a finite set called the set of states,
- Σ is a finite set called the input alphabet,
- $q_0 \in Q$, called the initial state,
- $F \subseteq Q$, called the set of final states, and
- $\delta : Q \times \Sigma \longrightarrow Q$ is a function called the transition function, see [6].

The remainder of this paper is organized as follows. In Section 2, some mathematical preliminaries. In Section 3, the purpose of this study is to determine some conditions on the transition function of A that ensure the existence of some properties. We give also a specific equivalence relation on set Q . Finally, we draw our conclusions in Section 4.

2. PRELIMINARIES

A monoid is a set M equipped with an associative product $x, y \mapsto xy$, together with a (left and right) unit 1. In the commutative case, it is common to use the additive notation: $x + y$ instead of xy and 0 instead of 1, see [Bil14, MS14] . If $X \subset M$, we write X^* for the submonoid of M generated by X , that is the set of finite products $x_1 x_2 \dots x_n$ with $x_1, x_2 \dots, x_n \in X$, including the empty product 1. It is the smallest submonoid of M containing X, see [Bil14, MS14] . Let Σ be a set, with we call an alphabet. A word w on the alphabet Σ is a finite sequence of elements of Σ, $w = (a_1, a_2, \dots, a_n)$ $a_i \in \Sigma,\ 1 \leq i \leq n.$ The set of all words on the alphabet Σ is denoted by Σ^* and is equipped with the associative operation defined by the concatenation of two sequences

$$(a_1, a_2, \dots, a_n)(b_1, b_2, \dots, b_m) = (a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_m)$$

This operation is associative. This allows us to write $w = a_1 a_2 \dots a_n$. The string consisting of zero letters is called the empty word, written ε. Thus, ε, 0, 1, 011, 1111 are words over the alphabet $\{0, 1\}$. The set Σ^* of words is equipped with the structure of a monoid. The monoid Σ^* is called the free monoid on Σ .

The reverse of a word $w = a_1 a_2 \dots a_n$, is $w^{-1} = a_n a_{n-1} \dots, a_1$.

Note that for all $u, v \in \Sigma^*, (uv)^{-1} = v^{1} u^{-1}$. The length of a word u , in symbols $|u|$, is the number of letters in u when each letter is counted as many times as it occurs. Again by definition, |ε|=0. The length function possesses some of the formal properties of Logarithm: $|uv| = |u| + |v|, |u^i| = i|u|$, for any words u and v and integers $i \geq 0$. For example $|011| = 3$ and $|1111| = 4$. For a subset B of Σ , we let $|w|_B$ denote the number of letters of w which are in B . Thus $|w| = \sum_{a \in \Sigma} |w|_a$. A language L over Σ^* is any subset of Σ^*, i.e, $L \subseteq \Sigma^*$.

A mapping $h : \Sigma^* \longrightarrow \Delta^*$, where Σ and Δ are alphabets, satisfying the condition $h(uv) = h(u)h(v)$, for all words u and v of Σ^*

is called a morphism, define a morphism h, it suffices to list all the words $h(a)$, where a ranges over all the (finitely many) letters of Σ. If M is a monoid, then any mapping $f:\Sigma \to M$ extends to a unique morphism $\hat{f}: \Sigma^* \to M$ For instance, if M is the additive monoid ℕ, and f is defined by $f(a) = 1$ for each $a \in \Sigma$, then $\hat{f}(u)$ is the length $|u|$ of the word u.

A binary relation on Σ^* is a subset $R \subseteq \Sigma^* \times \Sigma^*$. If $(x,y) \in R$, we say that x is related to y by R, denoted xRy. A binary relation R on a set Σ^* is said to be

- Reflexive if xRx for all x in Σ^*;
- Symmetric if xRy implies yRx;
- Transitive if xRy and yRz imply xRz.

The relation R is called an equivalence relation if it is reflexive, symmetric, and transitive. And in this case, if xRy, we say that x and y are equivalent. The set of all equivalence classes is denoted by Σ^*/R and is called the quotient of Σ^* mod R, see [Mar05].

A deterministic finite automata (DFA) is a quintuple $A = (Q, \Sigma, \delta, q_0, F)$, where

- Q is a finite set called the set of states,
- Σ is a finite set called the input alphabet,
- $q_0 \in Q$, called the initial state,
- $F \subseteq Q$, called the set of final states, and
- $\delta: Q \times \Sigma \to Q$ is a function called the transition function.

The extended transition function $\breve{\delta}: Q \times \Sigma^* \to Q$ is defined recursively as follows: for all $q \in Q$, $w \in \Sigma^*$ and $a \in \Sigma$, $\breve{\delta}(q,\varepsilon) = \varepsilon$ and $\breve{\delta}(q,wa) = \delta(\breve{\delta}(q,w),a)$.

3. SOME PROPERTIES ON THE DETERMINISTIC FINITE AUTOMATA

In this section we determine some conditions on the transition of a deterministic finite automata that ensure the existence of some properties.

Proposition 3.1

Let $A = (Q, \Sigma, \delta, q_0, F)$ be a deterministic finite automata, we have,

1. For all $q \in Q$, the set $Stab(q) = \{w \in \Sigma^*: \breve{\delta}(q,w) = q\}$ is a submonoid of free monoid $(\Sigma^*, \cdot, \varepsilon)$.
2. For all $q_1 \in Q$, $q_2 \in Q$, $w \in \Sigma^*$, if $\breve{\delta}(q_1, w) = q_2$ and $\breve{\delta}(q_2, w^{-1}) = q_1$, then,
   - $wStab(q_2)w^{-1} \subseteq Stab(q_1)$.
   - $u \in Stab(q_1) \Longrightarrow w^{-1}uw \in Stab(q_2)$.
3. For all $q \in Q$, the relation R_q defined over Σ^* by : $uR_qv \Leftrightarrow Stab(q)u = Stab(q)v$ is an equivalence relation.
4. $uR_qv \Leftrightarrow \exists x,y \in Stab(q): u = xv$ and $v = yu$.
5. The mapping $\psi: \Sigma^*/R_q \to O_q$, $\bar{u} \mapsto \breve{\delta}(q,u)$ is well defined, where $O_q = \{\breve{\delta}(q,u), u \in \Sigma^*\}$, and $\bar{u}$ the equivalence class of u with respect to R_q.

Proof

1. Let $q \in Q$, we have $\varepsilon \in Stab(q)$ because $\breve{\delta}(q,\varepsilon) = q$. Furthermore if $u, v \in Stab(q)$, i.e, $\breve{\delta}(q,u) = q$ and $\breve{\delta}(q,v) = q$, then $\breve{\delta}(q,uv) = \breve{\delta}(\breve{\delta}(q,u),v) = \breve{\delta}(q,v) = q$, then $uv \in Stab(q)$, therefore $Stab(q)$ is a submonoid of free monoid $(\Sigma^*, \cdot, \varepsilon)$.
2. Let $u \in wStab(q_2)w^{-1}$, i.e, there exists $x \in Stab(q_2)$ such that $u = wxw^{-1}$, we have
   $\breve{\delta}(q_1,u) = \breve{\delta}(q_1, wxw^{-1}) = \breve{\delta}(\breve{\delta}(q_1,w), xw^{-1}) = \breve{\delta}(q_2, xw^{-1})$
   $= \breve{\delta}(\breve{\delta}(q_2,x), w^{-1}) = \breve{\delta}(q_2, w^{-1}) = q_1$, then $u \in Stab(q_1)$.
   Let $u \in Stab(q_1)$, i.e, $\breve{\delta}(q_1,u) = q_1$, we have,
   $\breve{\delta}(q_2, w^{-1}uw) = \breve{\delta}(\breve{\delta}(q_2,w^{-1}), uw) = \breve{\delta}(q_1, uw)$
   $= \breve{\delta}(\breve{\delta}(q_1,u), w) = \breve{\delta}(q_1,w) = q_2$, then $w^{-1}uw \in Stab(q_2)$.
3. Let $q \in Q$, we have for all $u \in \Sigma^*$, $Stab(q)u = Stab(q)u$, then R_q is reflexive.
   The relation R_q is symmetric because for all $u, v \in \Sigma^*$, we have, $Stab(q)u = Stab(q)v$ implies $Stab(q)v = Stab(q)u$. Finally R_q is transitive because for all $u, v, t \in \Sigma^*$ we have,
   If $Stab(q)u = Stab(q)v$ and $Stab(q)v = Stab(q)t$, then $Stab(q)u = Stab(q)t$.
4. If uR_qv, then $Stab(q)u = Stab(q)v$, we have $u \in Stab(q)u$ because $u = \varepsilon u$ and

$\varepsilon \in \mathrm{Stab}(q)$, since $\mathrm{Stab}(q)u = \mathrm{Stab}(q)v$, then there exists $x \in \mathrm{Stab}(q)$ such that $u = xv$. A similar argument show that there exists $y \in \mathrm{Stab}(q)$ such that $v = yu$.

And there exists $x, y \in \mathrm{Stab}(q): u = xv \text{ and } v = yu$, we show that $\mathrm{Stab}(q)u = \mathrm{Stab}(q)v$,

let $w \in \mathrm{Stab}(q)u$, then there exists $\alpha \in \mathrm{Stab}(q)$ such that $w = \alpha u$, since $u = xv$, then $w = \alpha(xv) = (\alpha x)v$, we have $\alpha x \in \mathrm{Stab}(q)$, then $w \in \mathrm{Stab}(q)v$, finally $\mathrm{Stab}(q)u \subseteq \mathrm{Stab}(q)v$.

A similar argument shows that $\mathrm{Stab}(q)v \subseteq \mathrm{Stab}(q)u$. Therefore $\mathrm{Stab}(q)u = \mathrm{Stab}(q)v$.

5. It suffices we show that for all $\bar{u}, \bar{v} \in \Sigma^* / R_q : \bar{u} = \bar{v} \Rightarrow \breve{\delta}(q, u) = \breve{\delta}(q, v)$.

If $\bar{u} = \bar{v}$, then there exists $x, y \in \mathrm{Stab}(q): u = xv \text{ and } v = yu$, we have,

$\breve{\delta}(q, u) = \breve{\delta}(q, xv) = \breve{\delta}\left(\breve{\delta}(q, x), v\right) = \breve{\delta}(q, v)$.

Example 3.2

Let $A = (Q, \Sigma, \delta, q_0, F)$ be a deterministic finite automata, with $Q = \{q_1, q_2, q_3, q_4\}$, $\Sigma = \{0,1\}$, $q_0 = q_1$, $F = \{q_1\}$, $\delta(q_1, 1) = q_2$,

$\delta(q_1, 1) = q_2, \delta(q_2, 0) = q_3, \delta(q_3, 0) = q_4, \delta(q_4, 1) = q_1.$

We have $Stab(q_1) = (1001)^*$, $Stab(q_3) = (0110)^*$

And

$(10)\, Stab(q_3)(01) = (10)\,(0110)^*(01) \subseteq Stab(q_1).$

In following proposition, we define an equivalence relation on the set of states of a deterministic finite automata.

Proposition 3.3

Let $A = (Q, \Sigma, \delta, q_0, F)$ be a deterministic finite automata, the relation R on Q defined by:

$q_1 R q_2 \Leftrightarrow \exists w \in \Sigma^*: \breve{\delta}(q_1, w) = q_2 \text{ and } \breve{\delta}(q_2, w^{-1}) = q_1$

is an equivalence relation.

Proof

- For each $q_1 \in Q$, there exists $\varepsilon \in \Sigma^*$ such that $\breve{\delta}(q_1, \varepsilon) = q_1$, then by definition of R, $q_1 R q_2$, hence R is reflexive.
- If $q_1 R q_2$, then there exists $w \in \Sigma^*: \breve{\delta}(q_1, w) = q_2 \text{ and } \breve{\delta}(q_2, w^{-1}) = q_1$, then $\breve{\delta}(q_2, w^{-1}) = q_1$

and $\breve{\delta}(q_1, w) = q_2 = \breve{\delta}(q_1, (w^{-1})^{-1}) = q_2$. By definition of R, $q_2 R q_1$. Thus R is symmetric.

- If $q_1 R q_2 \text{ andIf } q_2 R q_3$, then there exists $w_1, w_2 \in \Sigma^*: \breve{\delta}(q_1, w_1) = q_2 \text{ and } \breve{\delta}(q_2, w_1^{-1}) = q_1$

and $\breve{\delta}(q_2, w_2) = q_3 \text{ and } \breve{\delta}(q_3, w_2^{-1}) = q_2$.

We have $\breve{\delta}(q_1, w_1 w_2) = q_3$ and $\breve{\delta}(q_3, (w_1 w_2)^{-1}) = q_1$. Thus $q_1 R q_3$. Therefore R is transitive.

Example 3.4

Let $A = (Q, \Sigma, \delta, q_0, F)$ be a deterministic finite automata, with $Q = \{q_1, q_2, q_3, q_4\}$, $\Sigma = \{0,1\}$, $q_0 = q_1$, $F = \{q_1\}$, $\delta(q_{1,1}) = q_2$,

$\delta(q_{1,1}) = q_2, \delta(q_{2,1}) = q_3, \delta(q_{3,1}) = q_4, \delta(q_{4,1}) = q_1.$

We have $\breve{\delta}(q_1, 11) = q_3 \text{ and } \breve{\delta}(q_{3,11}) = q_1$, then $q_1 R q_3$.

4. CONCLUSION

In this paper, we have determined some conditions on the transition function of A that ensure the existence of some properties. We give also a specific equivalence relation on set Q.

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