NUMERICAL RESULTS OF SOME INITIAL AND BOUNDARY VALUE PROBLEMS IN MECHANICS

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ABSTRACT: In this research article, the Variational Iteration method coupled with the polynomial approximation is used to find numerical solution to some homogenous and non-homogenous ordinary differential equations arising in mechanics.
***KEYWORDS**:* variational iteration method, boundary value problem, polynomial.

1. INTRODUCTION

Many researchers have used Variation Iteration Method (VIM) to find numerical solutions to different types of differential equations. In [OGA10], Olayiwola et al. proposed a modification of VIM to improve its convergence. The Variational Iteration Method was proposed by He J. H. [He99, He00, He06, He07] and has been successfully applied to solve many types of problems that arise in science and engineering [Alq11, Ola16a, Ola16b, Ola14].
In this study, the analytical and numerical solutions of fourth order homogenous and non-homogenous linear boundary value problems were investigated using a polynomial form of initial guess with the variational iteration method. We also make with the exact solution. We show that the method is a powerful technique for solving higher order boundary value problems.

2. VARIATIONAL ITERATION METHOD

The basic idea of application of general Lagrange multiplier and variational calculus was first proposed in 1978 by Inokuti et al [ISM78] in his method to solve nonlinear problems. Ji-Huan He modified the method of Inokuti and proposed the Variational Iteration Method (VIM). The idea of this method is constructing a correction functional by a general Lagrange multiplier.
The initial approximation can be freely chosen with possible unknown which can be determined by imposing the boundary and initial conditions.
To understand the basic procedure, we consider the following general differential equation.

$$Lu(s) + Nu(s) = f(s) \tag{1}$$

where L is the linear operator, N is the non-linear operator and $f(s)$ is an inhomogeneous term, according to the VIM ([He06, He07, Ola16a, Ola16b]).
The sequence $\{U_n\}$ can be constructed such that it converges to the exact solution through a correction functional as follows:

$$U_{n+1}(s) = U_n(s) + \int_o^s \lambda(\tau)\left[LU_n(\tau) + N\left(\breve{U}_n(\tau)\right) - f(\tau)\right]d\tau \tag{2}$$

where λ is the general Lagrangian multiplier which can be optimally identified by the Variational theory ([He06]), the subscript n denotes the nth order approximation: the equation (2) is assumed stationary when $\delta\breve{u}_n = 0$. After some iterations $U_n(s)$, $n \geq 0$ approaches the exact solution i.e.

$$U(s) = \lim_{n\to\infty} U_n(s) \quad (3)$$

In this paper, we assumed the initial approximation of the form:

$$U_0(s) = \sum_{i=0}^{k-1} a_i x^i \quad (4)$$

where k is the order of the differential equation.

3. APPLICATION OF THE METHOD

Example 1: We consider the fourth order linear inhomogeneous differential equation ([TO11]):

$$\frac{d^4 y}{dx^4} - 2\frac{d^2 y}{dx^2} + y = -8e^x \quad (5)$$

With the boundary conditions of

$$\begin{aligned} &y(0) = y''(0) = 1 \\ &y'(1) = y''(1) = -e \end{aligned} \quad (6)$$

Applying equ. (2) and (4) in (5) we obtained:

$$y_{n+1}(x) = y_n(x) + \int_0^s \lambda(\tau)\left[y_n^{iv}(\tau) - 2y_n''(\tau) + y_n(\tau) + 8e^{\tau}\right]d\tau. \quad (7)$$

Where

$$y_0(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 \quad (8)$$

$$\lambda(\tau) = \frac{(\tau - x)^{k-1}}{(k-1)!} \quad (9)$$

After seven iterations using Maple 18, we obtained:

$$
\begin{aligned}
y(x) =\, & a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \left(\frac{1}{6}a_2 - \frac{1}{24}a_0 - \frac{1}{3}\right)x^4 + \left(\frac{1}{10}a^3 - \frac{1}{120}a_1 - \frac{1}{15}\right)x^5 + \left(\frac{1}{120}a_2 - \frac{1}{360}a_0 - \frac{1}{30}\right)x^6 \\
& + \left(\frac{1}{280}a_3 - \frac{1}{2520}a_1 - \frac{1}{210}\right)x^7 + \left(-\frac{1}{840} + \frac{1}{5040}a_2 - \frac{1}{13440}a_0\right)x^8 + \left(-\frac{1}{7560} + \frac{1}{15120}a_3 - \frac{1}{120960}a_1\right)x^9 \\
& + \left(\frac{1}{907200}a_0 - \frac{1}{226800} - \frac{1}{1814400}a_2\right)x^{10} + \left(\frac{1}{9979200}a_1 - \frac{1}{2494800} - \frac{1}{604800}a_3\right)x^{11} \\
& + \left(-\frac{1}{479001600}a_0 + \frac{1}{39916800}a_2 - \frac{1}{19958400}\right)x^{12} + \left(\frac{1}{172972800}a_3 - \frac{1}{6227020800}a_1 - \frac{1}{259459200}\right)x^{13} \\
& + \left(-\frac{1}{4358914560\ 0}a_2 - \frac{1}{3632428800}\right)x^{14} + \left(-\frac{1}{2179457280\ 00}a_3 - \frac{1}{5448642000}\right)x^{15} \\
& - \frac{1}{8717829120\ 00}x^{16} - \frac{1}{1482030950\ 4000}x^{17} - \frac{1}{2667655710\ 72000}x^{18} - \frac{1}{5068545850\ 368000}x^{19} - \\
& \frac{1}{1013709170\ 07360000}x^{20} - \frac{1}{2128789257\ 154560000}x^{21} - \frac{1}{4683336365\ 7400320000}x^{22} \\
& - \frac{1}{1077167364\ 1202073600\ 00}x^{23} - \frac{1}{2585201673\ 8884976640\ 000}x^{24} - \frac{1}{6463004184\ 7212441600\ 0000}x^{25} \\
& - \frac{1}{1680381088\ 0275234816\ 000000}x^{26} - \frac{1}{4537028937\ 6743134003\ 2000000}x^{27} \\
& - \frac{1}{1270368102\ 5488077520\ 89000000}x^{28} - \frac{1}{3684067497\ 3915424810\ 5984000000}x^{29}
\end{aligned}
\tag{10}
$$

Imposing the initial and boundary conditions (6), we obtained the following system of algebraic equations.

$$a_0 = 0 \tag{11}$$

$$2a_2 = 0 \tag{12}$$

$$
\left\{\frac{457677527}{479001600}a_0 + \frac{881801009}{889574400}a_1 + \frac{5122563154\ 7}{4358914560\ 0}a_2 + \frac{2405327336\ 99}{2179457280\ 00}a_3 - \frac{1618863452\ 8389987243\ 983980979}{3684067497\ 3915424810\ 598400000}\right\} = 0 \tag{13}
$$

$$
\left\{-\frac{2131561}{3628800}a_0 - \frac{7341410}{39916800}a_1 + \frac{1020409631}{239500800}a_2 + \frac{30774997}{3773952}a_3 - \frac{7497408555\ 6445575457\ 313137}{1134257234\ 4185783500\ 840000}\right\} = -4e \tag{14}
$$

Solving equations (11-14) simultaneously for a_i, $i = 0,1,2,3$. Then

$$a_0 = 0, \tag{15}$$

$$a_1 = -\frac{2128727083\ 7082439005\ 0734814334\ 3744463141}{4752352419\ 3217931961\ 2003448661\ 215820800} + \frac{9985348905\ 4968929280\ 0}{1874295930\ 5622615606\ 49}e \tag{16}$$

$$a_2 = 0,$$

$$a_3 = \frac{3458335690\ 2692445161\ 2926567534\ 80108723}{4320320381\ 2016301782\ 9094044237\ 46892800} - \frac{8968622266\ 6898457600\ 0}{1874295930\ 5622615606\ 49}e \tag{17}$$

The analytical solution of eqn. (5) is in [TO11] as:

$$y(x) = x(1-x)e^x$$

Table 1: Numerical results of equs. (5-6)

X	Exact solution	Numerical Solution	Error
0	0	0	0.000E+00
0.1	0.099465383	0.099489031	2.365E-05
0.2	0.195424441	0.195470331	4.589E-05
0.3	0.28347035	0.283535633	6.528E-05
0.4	0.358037927	0.35811826	8.033E-05
0.5	0.412180318	0.412269774	8.946E-05
0.6	0.437308512	0.437399513	9.100E-05
0.7	0.422888069	0.422971413	8.334E-05
0.8	0.356086549	0.356151772	6.522E-05
0.9	0.22136428	0.221400812	3.653E-05
1	3.0179E-16	2.66E-10	2.659E-10

Example 2: We consider the fourth order linear homogeneous differential equation ([TO11]):

$$\frac{d^4y}{dx^4} - 2\frac{d^3y}{dx^3} + \frac{d^2y}{dx^2} = 0 \tag{18}$$

With the boundary conditions of

$$\begin{aligned} y(0) &= y''(0) = 1 \\ y'(1) &= y''(1) = e \end{aligned} \tag{19}$$

Applying equ. (2) and (4) in (18) we obtained:

$$y_{n+1}(x) = y_n(x) + \int_0^s \lambda(\tau)\left[y_n^{iv}(\tau) - 2y_n^{111}(\tau) + y_n^{11}(\tau)\right]d\tau. \tag{20}$$

Where

$$y_0(x) = a_0 + a_1x + a_2x^2 + a_3x^3 \tag{21}$$

After seven iterations using Maple 18, we obtained:

$$\begin{aligned} y(x) &= a_0 + a_1x + a_2x^2 + a_3x^3 + \left(-\frac{1}{12}a_2 + \frac{1}{2}a_3\right)x^4 + \left(\frac{3}{20}a_3 - \frac{1}{30}a_2\right)x^5 \\ &+ \left(-\frac{1}{120}a_2 + \frac{1}{30}a_3\right)x^6 + \left(\frac{1}{168}a_3 - \frac{1}{630}a_2\right)x^7 + \left(\frac{1}{1120}a_3 - \frac{1}{4032}a_2\right)x^8 \\ &+ \left(\frac{1}{8640}a_3 - \frac{1}{30240}a_2\right)x^9 + \left(\frac{19}{604800}a_2 - \frac{1}{5040}a_3\right)x^{10} + \left(\frac{67}{2217600}a_3 - \frac{1}{277200}a_2\right)x^{11} + \\ &\left(-\frac{1}{532224}a_3 + \frac{13}{79833600}a_2\right)x^{12} + \left(\frac{59}{1037836800}a_3 - \frac{1}{311351040}a_2\right)x^{13} \\ &+ \left(\frac{1}{4358914560\ 0}a_2 - \frac{1}{1210809600}a_3\right)x^{14} + \frac{1}{2179457280\ 00}a_3x^{15} \end{aligned} \tag{22}$$

Imposing the initial and boundary conditions (18), we obtained the following system of algebraic equations.

$$a_0 = 1 \tag{23}$$

$$2a_2 = 1 \tag{24}$$

$$a_0 + a_1 + \frac{3806028011\ 9}{4358914560\ 0} a_2 + \frac{3683553686\ 71}{2179457280\ 00} a_3 = e \tag{25}$$

$$\frac{682189}{239500800} a_2 + \frac{1691005648\ 9}{1037836800} a_3 = e \tag{26}$$

Solving equations (23-26) simultaneously for a_i, $i = 0,1,2,3$. We obtained:

$$a_0 = 1, \tag{27}$$

$$a_1 = -\frac{1588180149\ 5170700337\ 109}{1105642371\ 6048686976\ 000} + \frac{3182756494\ 019}{3551111862\ 690} e \tag{28}$$

$$a_2 = \frac{1}{2} \tag{29}$$

$$a_3 = -\frac{8868457}{1014603389\ 34} + \frac{1037836800}{1691005648\ 9} e \tag{30}$$

The analytical solution of eqn. (18) is in ([TO11])as:

$$y(x) = e^x$$

Table 2: Numerical results of equs. (18-19)

X	Exact solution	Numerical Solution	Error
0	1	1	0.000E+00
0.1	1.105170918	1.105170918	4.000E-11
0.2	1.221402758	1.221402757	9.800E-10
0.3	1.349858808	1.349858508	3.000E-07
0.4	1.491824698	1.491824694	4.000E-09
0.5	1.648721271	1.648721271	1.000E-11
0.6	1.8221188	1.8221188	1.000E-10
0.7	2.013752707	2.013752707	1.000E-10
0.8	2.225540928	2.225540928	4.000E-11
0.9	2.459603111	2.459603111	1.000E-11
1	2.718281828	2.718281828	4.590E-10

Example 3: We consider the fourth order linear homogeneous differential equation ([TO11]):

$$\frac{d^4 y}{dx^4} - y = 0 \tag{31}$$

With the boundary conditions of

$$\begin{aligned} y(0) &= y''(0) = 1 \\ y(1) &= y''(1) = 0 \end{aligned} \tag{32}$$

Applying equ. (2) and (4) in (31) we obtained:

$$y_{n+1}(x) = y_n(x) + \int_0^s \lambda(\tau)\left[y_n^{iv}(\tau) - y(\tau)\right] d\tau. \tag{33}$$

Where

$$y_0(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 \tag{34}$$

After seven iterations using Maple 18, we obtained:

$$\begin{aligned} y(x) &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \frac{1}{24} a_0 x^4 + \frac{1}{120} a_1 x^5 + \frac{1}{360} a_2 x^6 \\ &+ \frac{1}{840} a_3 x^7 + \frac{1}{40320} a_0 x^8 + \frac{1}{362880} a_1 x^9 + \frac{1}{181440} a_2 x^{10} + \frac{1}{6652800} a_3 x^{11} \\ &+ \frac{1}{479001600} a_0 x^{12} + \frac{1}{6227020800} a_1 x^{13} + \frac{1}{4358914560\ 0} a_2 x^{14} + \\ &\frac{1}{2179457280\ 00} a_3 x^{15} + \frac{1}{2092278988\ 8000} a_0 x^{16} + \frac{1}{3556874280\ 96000} a_1 x^{17} \\ &+ \frac{1}{3201186852\ 864000} a_2 x^{18} + \frac{1}{2027418340\ 1472000} a_3 x^{19} + \\ &\frac{1}{2432902008\ 176640000} a_0 x^{20} + \frac{1}{5109094217\ 1709440000} a_1 x^{21} \\ &+ \frac{1}{5620003638\ 8880384000} a_2 x^{22} + \frac{1}{4308669456\ 4808294400\ 00} a_3 x^{23} \end{aligned} \tag{35}$$

Imposing the initial and boundary conditions (32), we obtained the following system of algebraic equations.

$$a_0 = 1 \tag{36}$$

$$2a_2 = 1 \tag{37}$$

$$\begin{aligned} &\frac{2534333270\ 094778681}{2432902008\ 176640000} a_0 + \frac{5151684082\ 4285500441}{5109094217\ 1709440000} a_1 + \\ &\frac{5635617857\ 6807911956\ 1}{5620003638\ 8880384000\ 0} a_2 + \frac{3318307286\ 5759206567\ 7}{3314361120\ 3698688000\ 0} a_3 = 0 \end{aligned} \tag{38}$$

$$\begin{aligned} &\frac{2534333270\ 094778681}{1216451004\ 088320000} a_2 + \frac{5151684082\ 4285500441}{8515157028\ 6182824000\ 0} a_3 + \\ &\frac{3210080802\ 962401}{6402373705\ 728000} a_0 + \frac{4059664476\ 330413}{2432902008\ 1766400} a_1 = 0 \end{aligned} \tag{39}$$

Solving equations (36-39) simultaneously for a_i, $i = 0,1,2,3$. We obtained:

$$a_0 = 1, \tag{40}$$

$$a_1 = -\frac{8575039137\,1387259054\,8300734982\,8412509680\,44}{6530699693\,9368183952\,8961372930\,0897349000\,43}, \tag{41}$$

$$a_2 = \frac{1}{2}, \tag{42}$$

$$a_3 = -\frac{2858346379\,0462419684\,9216556379\,3328011317\,21}{1306139938\,7873636790\,5792274586\,0179469800\,86} \tag{43}$$

The analytical solution of eqn. (31) is in ([TO11]) as:

$$y(x) = \frac{1}{2Sinh(1)}\left(e^{1-x} - e^{x-1}\right)$$

Table 3: Numerical results of equs. (31-32)

X	Exact solution	Numerical Solution	Error
0	1	1	0.000E+00
0.1	0.873481691	0.873481691	5.404E-11
0.2	0.75570548	0.75570548	1.588E-10
0.3	0.645492624	0.645492624	8.215E-11
0.4	0.541740074	0.541740075	1.416E-10
0.5	0.443409442	0.443409442	1.496E-11
0.6	0.3495166	0.3495166	4.208E-11
0.7	0.259121838	0.259121838	8.907E-11
0.8	0.171320454	0.171320455	4.705E-10
0.9	0.085233703	0.085233703	4.870E-11
1	4.72354E-17	5.52E-11	5.516E-11

4. CONCLUSION

In this research article, we have successfully used the Variational Iteration Method coupled with the initial polynomial approximation to the fourth order boundary value problems. The results obtained were compared with the exact solution. It is apparently seen that the method is very powerful and efficient technique for finding the numerical solution to the fourth order boundary value problems.

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