PROVES OF THE EXISTENCE AND DETERMINATION OF BLOCKS SIZE IN PAIRWISE BALANCED DESIGNS

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ABSTRACT: A pairwise balanced design of index λ is a way of selecting blocks from a set of treatments (support set) such that any two treatments have covalency λ. If there are v treatments and if every block size is a member of some set K of positive integers, the design is designated a PB (v;K; λ). [Wal07]
Therefore, this research research work attempts to proof the existence of PBDs, shows the relationship that exists among the PBDs parameters (v, b, r, k, λ), and specify conditions for the construction of PBDs when $\lambda > 0$, k is odd and $\lambda = 1$.Thus, $v - k$ and $r - \lambda$ are proved to be nonnegative therefore, $r - k \geq 0$, and $b \geq v$. Also, a PBD(15; {4, 3}; 3) was constructed from the PBD(15; {5, 3}; 1).
KEYWORDS: Regular Design, Balanced Designs, Blocks Designs and Pairwise Balanced Designs.

THEOREM 1.1 [Wal07]
In any regular design,

$$bk = vr. \qquad (1.1)$$

Proof. One counts, in two different ways, all the ordered pairs (x, y) such that treatment x belongs to block y. Since every treatment belongs to r blocks, there are r ordered pairs for each treatment, so the number is vr. Similarly, each block contributes k ordered pairs, so the summation yields bk. Therefore bk = vr.
The number of blocks that contain a given treatment is called the replication number or frequency of that treatment. So the defining characteristics of a regular design are that all elements have the same replication number and that all blocks have the same size.
If all v treatments occur in a block of a design, that block is called complete. If a regular design has that property, then k = v and obviously, r = b. Such a design is a complete design. We say that a design is incomplete if at least one block is incomplete. If v = b, the design is called symmetric. If x and y are any two different treatments in an incomplete design, we shall refer to the number of blocks that contain both x and y as the covalency of x and y, and write it as λxy. Many important properties of block designs are concerned with this covalency function. In a pairwise balanced design, λ is sometimes called the index of the design. One often refers to a pairwise balanced design by using the five parameters (v, b, r, k, λ). [Hal35]
For example, the following blocks constitute a PBD (10, {3, 4}, 1).

{1,2,3,4} {2,5,8} {3,5,*10*} {4,5,9}

{1,5,6,7} {2,6,9} {3,6,8} {4,6,*10*}

{1,8,9,*10*} {2,7,*10*} {3,7,9} {4,7,8}.

Where, v = 10, b =3, k =4, λ =1, r = 3.{4}
If a PBD (v, {K}, λ) exists with b_i blocks of size k_i for each k_i, then the following expression connects PBD parameters

$$\lambda v\,(v-1) = \sum_i b_i k_i\,(k_i - 1).$$

Where,
v (order) is the size of V(elements of V are points, varieties or treatments)
b (block numbers) is the number of elements of B (elements of B blocks)
r (replication numbers) is the number of blocks to which every point belongs
k (blocks size) is the common size of each block and
λ (index) is the number of blocks to which every pair of distinct point belongs. {4}.
A balanced design with λ = 0, or a null design, is often called "trivial", and so is a complete design. We shall demand that a pairwise balanced design be not trivial; since completeness is already outlawed, the added restriction is that λ > 0. [Wal07]

THEOREM 1.2
In a (v, b, r, k, λ)-PBD,

$r(k-1) = \lambda(v-1)$. (1.2)

Proof. Consider the blocks of the design that contain a given treatment, x say. There are r such blocks. Because of the balance property, every treatment other than x must occur in λ of them. So, if we list all entries of the blocks, we list x, r times and we list every other treatment λ times.
A pairwise balanced design of index λ is a way of selecting blocks from a set of treatments (support set) such that any two treatments have covalency λ. [Wal07]
If there are v treatments and if every block size is a member of some set K of positive integers, the design is designated a PB(v;K; λ). So the parameters consist of two positive integers and one set of positive integers. The number of blocks is not normally treated as a parameter; one can have two pairwise balanced designs with the same parameters but with different numbers of blocks. Both

123, 145, 24, 25, 34, 35

and

123, 14, 15, 24, 25, 34, 35, 45

are PB(5; {3, 2}; 1)s, but they have six and eight blocks, respectively.
It should be observed that we do not require every member of K to be a block size. For instance, the two examples are also PB(5; {4, 3, 2}; 1)s. More generally, any PB(v;K; λ) is also a PB(v; L; λ) whenever K is a subset of L. Various results may be proven about pairwise balanced designs. For example, two copies of a PB(v;K; λ), taken together, constitute a PB(v;K;2λ). Just as obviously, there is a PB(v; {v}; λ) for all positive integers v and λ (it is the complete design with λ sets, each containing all v elements). The following theorem established existence of PBDs.

THEOREM 2.0
Suppose there exists a PB(v;K; λ), and for every element k of K there exists a PB(k; L; μ). Then there exists a PB(v; L; λμ).
Proof. Suppose we have a PB(v;K; λ) based on a v-set V . Suppose its blocks are $B_1, B_2, \ldots, B_n$, where B_i has k_i elements. We replace each block by a new collection of blocks. Given B_i, form a PB(k_i; L; μ), but instead of taking the numbers $1, 2, \ldots, k_i$ as treatments, use the elements of B This is done for every i. If x and y are any two treatments, then {x, y} was contained in λ of the B_i, and in each case B_i has been replaced by a collection of blocks, μ of which contain {x, y}. So x and y occur together λμ times in total. Therefore, the total collection of new blocks is a pairwise balanced design of index λμ, based on V . The size of any block of the new design is a member of L, so the design is a PB(v; L; λμ).{5}
For example, we can construct a PB(15; {4, 3}; 3) from the PB(15; {5, 3}; 1) with blocks

01234 56789 *ABCDE*
05*A* 06*C* 07*E* 08*B* 09*D*
16*B* 17*D* 18*A* 19*C* 15*E*
27*C* 2*SE*29*B* 25*D* 26*A*
38*D* 39*A*35*C* 36*E* 37*B*
49*E* 45*B* 46*D* 47*A* 48*C*.

We need a PB(5; {4, 3}; 3) and a PB(3; {4, 3}; 3). The former has blocks

1234, 1235, 1245, 1345, 2345 (2.1)

and the latter is the complete design 123, 123, 123. The block 01234 is replaced by a copy of the set of blocks (2.1) in which the treatments have been relabeled using the mapping

$(1, 2, 3, 4, 5) \rightarrow (0, 1, 2, 3, 4)$.

The new blocks are

0123 0124 0134 0234 1234. (2.2)

Similarly, 56789 and ABCDE are replaced by

5678 5679 5689 5789 6789 (2.3)

and

ABCD ABCE ABDE ACDE BCDE, (2.4)

respectively. The block 05A is replaced by

05*A* 05*A* 05*A*

and similarly for every other 3-block. So the required design consists of the blocks listed in (2.2), (2.3) and (2.4) and three copies of every 3-set in the original design.
The PB(15; {5, 3}; 1) that we used above is an example of a useful class of designs. We now give a more general construction.

THEOREM 2.1
There exists a PB(3k; {3, k}; 1) whenever k is odd.
Proof. We construct a design with treatment set $\{a_1, a_2, \ldots, a_k, b_1, b_2, \ldots, b_k, c_1, c_2, \ldots, c_k\}$. There are three blocks of size k (which we call “big blocks”), namely

$\{a_1a_2 \ldots a_k\}, \{b_1b_2 \ldots b_k\}, \{c_1c_2 \ldots c_k\}$.

The blocks of size 3 are the blocks $\{a_i b_i + x c_i - 2x : 1 \le i \le k,\ 1 \le x \le k\}$ where subscripts that exceed k are reduced modulo k. The pairs $\{a_i, a_j\}$, $\{b_i, b_j\}$, and $\{c_i, c_j\}$ occur once each, in the big blocks. The pair $\{a_i, b_j\}$ occurs in $\{a_i b_i + x c_i + 2x\}$ if and only if $j \equiv i + x \pmod{k}$; this will happen for only one value of x, either $x = j - i$ (if $j > i$) or $x = k - j - i$ (if $j \le i$). So only one block contains $\{a_i, b_i\}$. A similar remark applies to $\{b_i, c_j\}$. If $\{a_i, c_j\}$ occurs in $\{a_i b_i + x c_i + 2x\}$ then $i - 2x \equiv j \pmod{k}$, and this also uniquely defines x; since k is odd, the solution is $x \equiv 12 \cdot (k-1)(j-1) \pmod{k}$.
The construction above does not generalize to even values of k, because1/2 $(k - 1)(j - i)$ is not necessarily an integer in that case. However, we shall now prove anther theorem concerning the number of blocks in a pairwise balanced design.

LEMMA 1.1
In a pairwise balanced design with $\lambda = 1$, no two blocks have two common elements.
Proof. Suppose $B_1 = \{x, y, \ldots\}$ and $B_2 = \{x, y, \ldots\}$. Then$\{x, y\}$ is a subset of B_1, and also of B_2. So $\lambda_{xy} \ge 2$. But this contradicts the property that $\lambda = 1$.

THEOREM 2.2{5}
Suppose there is a P B(v; K ; 1) with b blocks, where $b > 1$. Then $b \ge v$. If $b = v$, then either the P B(v; K ; 1) has one block of size $v - 1$ and the rest of size 2, or else $b = v = k^2 - k + 1$ for some integer k and all the blocks have size k. [BE48]
Proof. Suppose a P B(v; K ; 1) has treatments $t_1, t_2, \ldots, t_v$ and b blocks $B_1, B_2, \ldots, B_b$. Say k_i is the number of elements in B_i, and that t_j belongs to r_j blocks. (We call r_j the frequency or replication number of t_j.) Then, counting all elements of all blocks,

$$\sum_{j=1}^{v} r_j = \sum_{i=1}^{b} k_i$$

If t does not belong to B_i, then t_j must belong to at least k_i blocks: For every element x of B_i there is a block that contains t and x, and these blocks are all disjoint because of Lemma 2.3. So,

$$t_j \notin B_i \Rightarrow k_i \le r_j . \qquad (2.6)$$

The blocks are incomplete, and there are no blocks of size 1. So

$$1 < k_i < v \text{ for } 1 \le i \le b. \qquad (2.7)$$

There must be some treatment whose replication number is minimal. Say it is t_v, and write $r_v = m$. Relabel the blocks so that those containing t_v are $B_1, B_2, \ldots, B_m$. We can select an element of each block, other than t_v, and clearly all these elements are different. Suppose (after relabeling) that ti ε B_i, $t_i \neq t_v$
Then if $1 \le i \le m, 1 \le j \le m$, and $i \neq j, t_j \notin B_i$;{1}
In particular, $t_1 \notin B_2$, $t_2 \notin B_2 \ldots\ldots t_m, t_m \notin B_1$ so, from 2.6

$$K_2 \le r_1,\ K_3 \le r_2, \ldots\ldots\ldots\ldots k_m \le r_{m-1}, k_1 \le r_m \qquad (2.8)$$

where

$$\sum_{i=1}^{m} ki \le \sum_{j=}^{m} rj . \qquad (2.9)$$

Also, $t_v \notin B_i$ for $i > m$, $k_i \le r_v$ for $i > m$, so

$$\sum_{i=m+1}^{b} k_i \le \sum_{j=m+1}^{b} rv \qquad (2.10)$$

Adding (2.9) and (2.10) and comparing with (2.5), we obtain

$$\sum_{j=1}^{v} rj = \sum_{j=1}^{b} ki = \sum_{i=1}^{m} ki + \sum_{i=1+1}^{b} ki \le \sum_{j=1}^{m} rj + \sum_{j=1}^{b} rv \le \sum_{j=1}^{b} rj$$

(since $r_v \le r_j$ for all j), and this is impossible if $b < v$ because the r_j are all positive. In particular, suppose $b = v$. Then each of the inequalities in (2.8) must be an equality, and also $k_i = r - i$ for all $i > m$. If we relabel the treatments $t_1, t_2, \ldots, t_m$, we obtain $r_i = k_i$, all $i \in \{1 \ldots v\}$ for some ordering of treatments and blocks. Moreover, $t_v$is unchanged. Let us further relabel the treatments (and simultaneously, the blocks) so that

$$r_1 \ge r_2 \ge \ldots \ldots\ldots\ldots\ldots \ge r_v .$$

(Since t_v had minimum frequency, it has still not been disturbed.) We consider the various possibilities.
(i) Suppose
$r_1 > r_2$. Then $r_1 > r_j$ for all $j \ge 2$. So $k_1 = r_1 > r_j$ ($j \ge 2$). From (2.6), $t_j \in B_1$ for all $j > 1$. Of course, $t_r \notin B_1$. So

$$B_1 = \{t_2, t_3, \ldots, t_v\},$$

and the other blocks must be

$$\{t_l, t_2\}, \{t_1, t_3\}, \ldots, \{t_2, t_v\}.$$

(ii) Suppose $r_1 = r_2 = \ldots = r_{j-1} > r_j$, where $j > 2$. Then $t_j \in B_1$ and $t_j \in B_2$ (from (2.6)); since $t_v \in B_1 \cap B_2$, the only possibility (according to Lemma 2.3) is $t_j = t_v$ and $j = v$. So we consider that case. Since $r_v < r_{v-1} = k_{v-1} < v$ (from (2.7)) there are at least two blocks not containing t_V.One might be B_V,but suppose the other is B_X , where x = v. Then from (2.6) we have $r_x = k_x \le r_v$, a contradiction. [Hal35]

(iii) Finally, suppose $r_1 = r_2 = \dots = r_V$. We have constant block size and constant frequency, a balanced incomplete block design. From (1.1) and (1.2) we immediately deduce that $b = v = k^2 - k + 1$, where k is the common block size.

We could generalize pairwise balanced designs by requiring that every set of t treatments occurs in a fixed number of blocks. This is called a t-wise balanced design. [Fis40]

To prove that b ≥ v when constructing pairwise balanced designs.

Proof: from (1.1) and (1.2) respectively. We then proceed: We assume we are given a (v,b,r,k,λ)-design with blocks $B_1,B_2,B_3\dots\dots B_n$. [Hal35]

Given n readings $y_1,y_2,y_3,\dots\dots y_n$,their mean $\bar{y}$ as $n^{-1}\sum \bar{y}_i$; and their variance is $v = n^{-1}\sum(y_i - \bar{y})^2$ is a non-negative. In order to reduce the size of numbers when calculating variances, the identity

$$\sum (y_i - \bar{y})^2 = \sum y_i^2 - n\,\bar{y}^2 \qquad (2.11)$$

is used. Now we write n=b-1. Define yi to be the size of the intersection $B_i \cap B_b$. We count the occurrence of pairs of numbers of Bb in the other blocks. Each pair has λ-1 further occurrences, so $\sum i/2\ y_i(y_i-1) = 1/2k(k-1)(\lambda-1)$ on the other hand, $\sum y_i = k(r-1)$

Whence we have

$$\bar{y} = (b-1)^{-1}k(r-1) \text{ and so}$$

$$\sum y_i^2 = k(k-1)(\lambda-1) + k(r-1)$$

Therefore,

$$\sum y_i^2 - (b-1)\bar{y}^2 k(k-1)(\lambda-1) + k(r-1) - (b-1)^{-1}k^2(r-1)^2$$

several identities are now used .The least obvious is the observation that

$$k^2(r-1)^2 - r^2(k-1)^2 = \{k(r-1) - r(k-1)\ (k(r-1) + r(k-1)\}$$
$$= (r-k)(2kr-k-r), \text{ so that}$$
$$k^2(r-1)^2 = r^2(k-1)^2 + (r-k)(2kr-k-r)$$
$$= \lambda(v-1)r(k-1) + (r-k)(2kr-k-r)$$

Using (1.2), others needed are:

$$k(b-1) = vr-k = r(v-1) + (v-k),$$
$$\lambda v = \lambda + vk - v$$

which follows from (1.1) and(1.2) respectively, we then proceed:

$$(b-1)\{\sum \bar{y}_i^2 - (b-1)\,\bar{y}_i^2\}$$
$$= (b-1)k(k-1)(\lambda-1) + (b-1)k(r-1) - k2(r-1)^2$$
$$= (b-1)k\,((k-1)(\lambda-1) + r - 1))$$
$$-(\lambda(v-1)r(k-1) + (r-k)(k + r - 2kr))$$
$$= r(v-1)(k\lambda - k - \lambda + r) + (r-k)(k\lambda - k - \lambda + r)$$
$$-\lambda(v-1)r(k-1) + (r-k)(k + r - 2kr)$$
$$= r(v-1)(k\lambda - k - \lambda + r - k\lambda + \lambda)$$
$$+(r-k)(k\lambda - k - \lambda + r + k + r - 2kr)$$
$$= r(v-1)(r-k) + (r-k)(k\lambda - \lambda + 2r - 2kr)$$
$$= (r-k)(rv + k\lambda - 2kr + r - \lambda)$$
$$= (r-k)(rv + k\lambda - kr - r(k-1) - \lambda)$$
$$= (r-k)(rv + k\lambda - kr - \lambda(v-1) - \lambda)$$
$$= (r-k)(rv + k\lambda - kr - \lambda v)$$
$$= (r-k)(v-k)(r-\lambda).$$

Now $v - k$ and $r - \lambda$ are nonnegative, so

$$r - k \ge 0,$$

and it follows from (1.1) that $b \ge v$.

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