CONSTRUCTION OF (k^2, b, r, k, 1) RESOLVABLE BALANCED INCOMPLETE BLOCK DESIGNS (RBIBD) USING NIM ADDITION TABLES OF ORDER 2^n, $2 \leq n \leq 5$

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ABSTRACT: Resolvable Balanced Incomplete Block Designs (RBIBDs) are important combinatorial designs that have useful applications in various fields of human endeavour. In this paper, Nim addition tables, from the game of Nim, of order 2^n, $2 \leq n \leq 5$ were used as the basis for the construction of some (k^2, b, r, k, 1) RBIBDs. Nim addition tables of order 2^n, $2 \leq n \leq 5$ were constructed. These tables were special Latin squares that obeyed the Finite Groups theory and closed *n*-nim-regularity conditions for closed *n*-nim-regular games. The Bose's generalized method of constructing Mutually Orthogonal Latin Squares (MOLS) was used to obtain 2^n-1 MOLS for each *n*. The MOLS were super-imposed on one another and successive diagonalization algorithm was used to obtain the RBIBDs from the super-imposed MOLS. The RBIBDs constructed were *(16, 20, 5, 4, 1)*, *(64, 72, 9, 8, 1)*, *(256, 272, 17, 16, 1)* and *(1024, 1056, 33, 32, 1)* RBIBDs. These RBIBDs are all in existence and a link was thus established between the game of Nim and RBIBDs.
KEYWORDS: Mutually Orthogonal Latin Squares; Successive diagonalization algorithm; Impartial Combinatorial games; *closed n*-nim-regular matrix; Finite groups.

1. INTRODUCTION

Balanced Incomplete Block Designs (BIBD) are groups of combinatorial designs that have been studied widely in literature. A BIBD is a pair (*V*, *B*) where *V* is a *v*-set and *B* is a collection of *b* *k*-subsets of *V* (blocks) such that each element of *V* is contained in exactly *r* blocks and any 2-subset of *V* is contained in exactly λ blocks. The numbers *v*, *b*, *r*, *k*, and λ are the parameters of the BIBD, ([MR07]). A **parallel class** in (*V*, *B*) is a subset of disjoint blocks from *B* whose union is *V*. A partition of *B* into *r* parallel classes is called a **resolution**, and (*V*, *B*) is said to be a **Resolvable BIBD** if *B* has at least one resolution, ([Sti04]).

Resolvable BIBDs have found useful applications in various fields of human endeavour including Agriculture, Cryptology, Secret Sharing Schemes, Sports Scheduling among others ([Sti04, Ote16, CC57, D+07, AF07]). In Agriculture, for instance, blocks are usually grouped into sets with each group containing each treatment once. This has a great advantage where it is feasible to run and analyze the design replicate by replicate or omit some replicates, ([CR00]).

Different construction techniques exist in the literature for the construction of RBIBDs. Some of these included the works of ([A+01, A+03]) and so on.

In this paper, we constructed some *(k^2, b, r, k, 1) RBIBDs* using Nim addition tables of order 2^n, $2 \leq n \leq 5$.

2. THE NIM GAME

Nim is a mathematical game belonging to the family of impartial combinatorial games. The theory underlying this game was presented in ([Bou01]). This combinatorial game involves two players, a set of positions and a set of legal moves between positions. It is usually played with heaps of coins, match sticks, stones and so on. Players alternate moves. Each player at his or her turn selects one of the heaps and removes at least one of the coins from the selected heap. The player may remove all the coins from the selected heap thereby leaving the heap empty (the heap is said to be out of play). The first player who cannot move anymore loses the game, ([MAA89]).

Nim is impartial because the winning positions and the available moves are the same for the two players; the set of allowable moves depends only on the position of the game and not on which of the two players is moving.

Variants of Nim abound in literature. Some of these can be found in ([LW14, LR16, Wyt07, Whi63, DH13]) among others.

The game of Nim is of central importance to the Sprague-Grundy theorem of impartial games, ([MAA89]. Also, it is connected to lexicographic codes ([CS86, KK02, Smi66, Fra96, D+09, Ric07]) Also, linear codes over Galois Fields have also been linked to fractional replication plans in statistical design of experiments, ([Bos61]).

In this paper, we constructed some $(k^2, b, r, k, 1)$ *RBIBDs* using Nim addition tables of order 2^n, $2 \le n \le 5$ thus establishing the relationship between the game of Nim and RBIBDs.

2.1 The Nim Addition Table

The Nim addition table consists of a field of numbers called *Nimbers*. This table is used to determine winning or losing positions in any game of Nim, thereby making it possible for us to predetermine the winner in a game. The Nim addition of two non-negative integers is their addition without carries in base 2. For instance, 13*15 = 1101+1111 = 0010 = 2.

2.1.1 Properties of the Nim Addition Table

If a and b are two integers, their Nim addition is another integer denoted by $a*b$. The properties of Nim addition table, according to ([MAA89]) are:

- Zero element: $0*a = a*0 = a$
- Commutativity: $a*b = b*a$
- Associativity: $[a*b]*c = a*[b*c]$
- The Nim addition of two different powers of two is the same as their ordinary sum e.g. $4*16 = 20$
- The Nim addition of two equal powers of two is zero e.g. $8*8 = 0$

Definition 1 ([B+10]):
Given that Z_n is the set (0, 1, 2, …., 2^n-1) of the 2^n first nonnegative integers and for any two values a, b ε Z_n, $a*b$ is the Nim addition value and $(Z_n,*)$ is a finite group making the addition table a Latin square.

Definition 2 ([B+10]):
A is a semi-infinite matrix defined by $A(a, b) = a*b$ for a, b ε $Z_{\geq 0}$ and A_n is the finite matrix consisting of the first 2^n rows and first 2^n columns of A.

Definition 3 ([D+09]):
For every positive integer n, the matrix A_n is *closed n-nim-regular*, implying that it is a Latin square over Z_n.

3. METHODOLOGY

Nim addition tables were constructed for 2^n, $2 \le n \le 5$. Based on *Definitions 1* to *3* above, these tables were *finite groups* and *closed n-nim-regular* matrices A_2, A_3, A_4 and A_5 over Z_n thus making them Latin squares over Z_n. $2^n - 1$ Mutually Orthogonal Latin Squares (MOLS) were obtained for each A_n using Bose's Generalized method of constructing MOLS, ([Joh71]). This involved cyclically rotating all the rows of A_n except the first. The MOLS obtained for each A_n were super-imposed on one another and successive diagonalization algorithm from ([KF79]) was used to obtain the RBIBDs from the super-imposed MOLS. These procedures were accomplished using MATLAB codes.

3.1 MATLAB Code for constructing $(k^2, b, r, k, 1)$ *RBIBD* from Nim addition table of order 2^n, $2 \le n \le 5$

Is presented in the following listing.

```
clear all
clc
n=input('enter n: ');
first=2^n;
second=first;
%% construct Nim addition table
for x=0:first
   for y=0:second
      fbin=dec2bin(x);% convert first no to base 2
      sbin=dec2bin(y);% convert second no to base 2
      lf=length(fbin);% find no of binary digits in first no
      ls=length(sbin);% find no of binary digits in second no
      if lf==ls
         n=lf;
      elseif lf~=ls % if no of bits used to represent both binary nos is no equal
         n=max(lf,ls); % determine which has the greater no of bits
      end
      fbin=dec2bin(x,n); %and represent first no
      sbin=dec2bin(y,n);%and second no using the same greater no of bits
      nimsum=zeros(1,n);
      if x~=0&&y~=0 % if neither of first and second nos is zero
         for i=1:n
            if fbin(i)==sbin(i)
```

```
            nimsum(i)=0;
          else
            nimsum(i)=1;
          end % nimsum is base 2 addition without carry
        end
        conv2str=mat2str(nimsum);% converts the binary addition to string format
        lc=length(conv2str);% find the no of elements in the string
        if (x==1&&y==1)||(x==0&&y==0)
          nimsum10=0;
        else
          nimsum10=bin2dec(conv2str(2:lc-1));%convert the nim addition to base 10
        end
      else
          nimsum10=x+y; %just add the other no to 0
      end
      nimsum101(x+1,y+1)=nimsum10;
   end
end
nimtable=nimsum101(1:first,1:second)
 %% generate Mutually Orthogonal Latin Square
d=nimtable;
[rw,cn]=size(d);
e=zeros(((rw-1)*cn),rw);k=1;
for i=1:rw-1
e(k:i*cn,:) = [d(1,:);circshift(d(2:cn,:),i)];
k=(i*cn)+1;
end
mols=e
%% construct RBIBDs
n=input('confirm the value of n: ')
bb=reshape(1:((2^n)*(2^n)),2^n,2^n)';
[rw1 cn1]=size(bb);
cc=bb';
rbibd=zeros((rw1*(rw1+1)),cn1);
count=1
rbibd1=bb
count=2
rbibd2=cc
rbibd=rbibd2;
% bibd3=zeros(rw,cn);
j=0;
count=3
for count=3:rw1+1
for i=1:rw1
   rbibdNEW(:,i)=circshift(rbibd(:,i),j);
   j=j-1;
end
rbibd=rbibdNEW
count=count+1
end
```

4. RESULTS

Results obtained when $n = 2$, and A_2 is the finite matrix consisting of the first 2^2 rows and first 2^2 columns of A are presented only in this paper to conserve space. Other results for $n = 3$, 4, and 5 can be easily obtained using the MATLAB Code presented in sub-section 3.1.

4.1 Nim addition table when $n = 2$ (Matrix A_2)

0	1	2	3
1	0	3	2
2	3	0	1
3	2	1	0

4.2 2^2-1 MOLS for A_2

(1)

0	1	2	3
3	2	1	0
1	0	3	2
2	3	0	1

(2)

0	1	2	3
2	3	0	1
3	2	1	0
1	0	3	2

(3)

0	1	2	3
1	0	3	2
2	3	0	1
3	2	1	0

4.3 Constructed RBIBD from super-imposed MOLS

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16

1	5	9	13
2	6	10	14
3	7	11	15
4	8	12	16

1	6	11	16
2	7	12	13
3	8	9	14
4	5	10	15

1	7	9	15
2	8	10	16
3	5	11	13
4	6	12	14

1	8	11	14
2	5	12	15
3	6	9	16
4	7	10	13

The parameters of the RBIBD constructed from A_2 are $v = k^2 = 16$, $b = 20$, $r = 5$, $k = 1$, $\lambda = 1$ while the summary of all results obtained for $2 \leq n \leq 5$ are presented in Table 1 below.

Table 1: Summary of Results Obtained

Nim Addition Table	***(k^2, b, r, k, 1)RBIBD***
A_2	*(16, 20, 5, 4, 1)RBIBD*
A_3	*(64, 72, 9, 8, 1)RBIBD*
A_4	*(256, 272, 17, 16, 1) RBIBD*
A_5	*(1024, 1056, 33, 32, 1) RBIBD*

5. CONCLUSION

Resolvable Balanced Incomplete Block Designs, with different construction techniques in literature, are important combinatorial designs that have useful applications in various fields of human endeavour. In this paper, Nim addition tables of order 2^n, $2 \leq n \leq 5$ were used as the basis for the construction of some $(k^2, b, r, k, 1)$ RBIBDs. Nim addition tables of order 2^n, $2 \leq n \leq 5$, which were shown to be special Latin squares, were constructed. 2^n-1 MOLS were obtained for each Latin square using Bose's Generalized method of constructing MOLS. The MOLS obtained for each square were super-imposed on one another and successive diagonalization algorithm was used to obtain the RBIBDs from the super-imposed MOLS. The RBIBDs constructed were *(16, 20, 5, 4, 1)*, *(64, 72, 9, 8, 1)*, *(256, 272, 17, 16, 1)* and *(1024, 1056, 33, 32, 1)* RBIBDs. These RBIBDs are all in existence thus establishing a link between the game of Nim and Resolvable Balanced Incomplete Block Designs.

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