DEVELOPMENT TEACHERS' MATHEMATICS COMPETENCY THROUGH TEACHING COMPLEX NUMBERS IN HIGH SCHOOL IN VIETNAM

Tran Trung Tinh

National Academy of Education Management, Hanoi city, Vietnam

Corresponding Author: Tran Trung Tinh, tinhtckh@gmail.com

ABSTRACT: In teaching mathematics at high schools, the mathematics competency of teacher is an important procedure in teaching process. This paper, I pointed out some application of complex numbers in mathematics. And i hope it will help the teachers to develop high school their mathematics competency in Vietnam.
KEYWORDS: Teachers' mathematics competency, complex numbers, teaching methods.

1. BUILDING A C FIELD

We review Carte $T = R\times R = \{(a,b) \mid a,b\in R\}$ and the following definition:

$(a,b) = (c,d)$ if and only if $a = c, b = d$

$$(a,b)+(c,d) = (a+c, b+d)$$

$$(a,b).(c,d) = (ac-bd, ad+bc).$$

We can write, $(a,b).(c,d)$ through $(a,b)(c,d).$ From the definition of multiplication:

- Where $i = (0,1)\in T$ we have $i^2 = i.i = (0,1)(0,1) = (-1,0)$
- $(a,b)(1,0) = (a,b) = (1,0)(a,b)$
- $(a,b) = (a,0)+(0,b) = (a,0)+(b,0)(0,1), \forall (a,b)\in T.$

Notation C is set T with Mathematical operations above. We get:

Lemma 1. *Map* $\phi: R\to C, a\mapsto (a,0),$ *Is a single map and such that* $\phi(a+a') = \phi(a)+\phi(a'), \phi(aa') = \phi(a)\phi(a')$ *forall* $a,a'\in R.$

$(a,0)\in C$ is the same as $a\in R.$ Then we have $(a,b) = (a,0)+(b,0)(0,1) = a+bi$ where $i^2 = (-1,0) = -1.$ Therefore i or a or $a+bi$ are equal in $C.$

Deduce $C = \{a+bi \mid a,b\in R, i^2 = -1\}$ and in C we have the following results:

$a+bi = c+di$ if and only if $a = c, b = d$

$$a+bi+c+di = a+c+(b+d)i$$

$$(a+bi)(c+di) = ac-bd+(ad+bc)i.$$

This objects $z = a+bi\in C$ considered as a *complex number* with a is the real part (Notation $Re(z),$) and b is the imaginary part, (Notation $Im(z)$); and i is the imaginary unit. Complex number $a-bi$ is called a conjugate complex of $z = a+bi$ and notation $\bar{z} = \overline{a+bi}.$ We have $z\bar{z} = (a+bi)(a-bi) = a^2+b^2, \overline{z_1z_2} = \overline{z_1}.\overline{z_2}$ and The modulus of a complex number z is $|z| = \sqrt{z\bar{z}}$. The argument number of a complex number $z' = c+di$ is $-z' = -c-di$, then we have $z-z' = (a+bi)-(c+di) = a-c+(b-d)i.$

Consider the system of coordinate $(Oxy).$ This Complex number $z = a+bi$ we get a point $M(a;b).$ and it is a bijection

$$C\to R\times R$$
$$z = a+bi\mapsto M(a;b).$$

In mathematics, the complex plane or z-plane is a geometric representation of the complex numbers established by the real axis and the perpendicular imaginary axis. with the real part of a complex number represented by a displacement along the x-axis, and the imaginary part by a displacement along the y-axis.

In geometry, a seventeen-sided polygon is a constructible polygon (that is, one that can be constructed using a compass and unmarked straightedge). This was shown by Carl Friedrich Gauss in 1796 at the age of 19.

Proposition 1. R *field is in* C *field*

Proof: We see that C is a commutative ring, where 1 is unit. Suppose $z = a+bi\neq 0.$ then $a^2+b^2>0.$ Suppose $z' = x+yi\in C$ must satisfy requirements $zz' = 1$ or $\begin{cases} ax-by = 1 \\ bx+ay = 0. \end{cases}$ We get $x = \dfrac{a}{a^2+b^2}, y = -\dfrac{b}{a^2+b^2}.$ Deduce $z' = \dfrac{a}{a^2+b^2}-\dfrac{b}{a^2+b^2}i$ is a inverse number of $z.$

Deduce C is a field. Map $C\to C, z\mapsto \bar{z},$ is a conjugate self- isomorphism. Homogeneoust $a\in R$

with $a+0i \in C$ and we can say that $R \subset C$.
Notation that, inverse number of $z \neq 0$ is $z^{-1} = \dfrac{\overline{z}}{|z|^2}$ và $\dfrac{z'}{z} = z'z^{-1} = \dfrac{z'\overline{z}}{|z|^2}$.

Definition 1. [N+15] *Given complex number* $z \neq 0$. *Suppose* M *is a point in the complex plane for expression* z. *The angle measurement created by first ray* Ox *and last ray* OM *called Argument of* z *and notation* $Arg(z)$. *The angle* $\alpha = \widehat{xOM}, -\pi \leq \alpha \leq \pi$, *called argument of* z *and notation* $Argz$. *Argument of 0 complex is not definition.*

Notation that, if α is a argument of z then forall argument of z are $\alpha + k.2\pi$ where $k \in Z$. If $z \neq 0$, noindent $\alpha + k.2\pi$ is Argument of z.

Notation $r = \sqrt{z\overline{z}}$. Therefore, the complex number $z = a+bi$ has $a = r\cos\alpha, b = r\sin\alpha$. Deduce, if $z \neq 0$ then $z = r(\cos\alpha + i\sin\alpha)$, we called *dạng lượng giác* of z.

Proposition 2. *Given complex number* z_1, z_2 *and* $z_1 = r_1(\cos\alpha_1 + i\sin\alpha_1)$, $z_2 = r_2(\cos\alpha_2 + i\sin\alpha_2)$, $r_1, r_2 \geq 0$, *we always have*

- $|z_1z_2| = |z_1||z_2|$, $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$ và $|z_1+z_2| \leq |z_1| + |z_2| = r_1 + r_2$.
- $z_1z_2 = r_1r_2[\cos(\alpha_1+\alpha_2) + i\sin(\alpha_1+\alpha_2)]$
- $\dfrac{z_1}{z_2} = \dfrac{r_1}{r_2}[\cos(\alpha_1-\alpha_2) + i\sin(\alpha_1-\alpha_2)]$ where $r_2 > 0$.

Proof: It's easier to get the result.

Example 1. *With* $a+bi = (x+iy)^n$ *we have* $a^2+b^2 = (x^2+y^2)^n$.

Proof: From $a+bi = (x+iy)^n$ deduce $a-bi = (x-iy)^n$. Therefore $a^2+b^2 = (x^2+y^2)^n$.

Proposition 3. [Moivre] *If* $z = r(\cos\alpha + i\sin\alpha)$ *then forall* $n \in Z^+$ *we have*
$z^n = r^n[\cos(n\alpha) + i\sin(n\alpha)]$.

Proof: Using Mathematical induction methods by n to get the result.

Corollary 1. *An* n *th root of a complex number* $z = r(\cos\alpha + i\sin\alpha) \neq 0$ *are different values in*
$z_k = r^{1/n}(\cos\dfrac{\alpha+2k\pi}{n} + i\sin\dfrac{\alpha+2k\pi}{n})$ *where* $k = 1,2,\ldots,n$.

Next, we prove a Euler's famous theorem below:

Theorem 1. [Euler] *For all* x *we have* $e^{ix} = \cos x + i\sin x$.

Proof: From
$e^{ix} = 1 + \dfrac{ix}{1!} + \dfrac{(ix)^2}{2!} + \dfrac{(ix)^3}{3!} + \cdots + \dfrac{(ix)^n}{n!} + \cdots$ we deduce

$$e^{ix} = (1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^n\frac{x^{2n}}{(2n)!} + \cdots)$$
$$+ i(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^{n-1}\frac{x^{2n-1}}{(2n-1)!} + \cdots).$$

Therefore $e^{ix} = \cos x + i\sin x$.

Corollary 2. *For all Real numbers* x, y *we always have*

- $e^{ix}e^{iy} = e^{i(x+y)}$ and $(e^{ix})^n = e^{inx}$ forall $n \in Z$.
- $\dfrac{e^{ix}}{e^{iy}} = e^{i(x-y)}$ and $\overline{e^{ix}} = e^{-ix} = \dfrac{1}{e^{ix}}$.
- $\cos x = \dfrac{e^{ix}+e^{-ix}}{2}, \sin x = \dfrac{e^{ix}-e^{-ix}}{2i}$.
- $< e^{ix}, e^{iy} > = \dfrac{1}{2}(\dfrac{e^{ix}}{e^{iy}} + \dfrac{e^{iy}}{e^{ix}}) = \cos(x-y)$.

Proof: From theorem 1, we easier to get the result.

Given three distinctive points A, B, C corresponding to three complex numbers a, b, c.

We notation $[A,B,C] = \dfrac{c-a}{c-b} \in C$ and we call it *single ratio* of point sets A, B, C. We have result.

Corollary 3. *Given three distinctive points* A, B, C *corresponding to three complex numbers* a, b, c *all on a line if and only if* $[A,B,C] = \dfrac{c-a}{c-b} \in R$.

Proof: Set $c-a = re^{i\alpha}, c-b = r'e^{i\beta}$. then we have $[A,B,C] = \dfrac{c-a}{c-b} = \dfrac{r}{r'}e^{i(\alpha-\beta)}$. three points A, B, C all on a line if and only if $\arg\dfrac{c-a}{c-b} = k\pi$ or $\dfrac{r}{r'}e^{ik\pi} = \pm\dfrac{r}{r'}$. Therefore, three distinctive points A, B, C corresponding to three complex numbers a, b, c all on a line if and only if $[A,B,C] = \dfrac{c-a}{c-b} \in R$.

Dot product and *deviation product* of two complex numbers z_1, z_2, notation $< z_1, z_2 >$ and $[z_1, z_2]$, It is defined as following:

$< z_1, z_2 > = \frac{1}{2}(z_1\overline{z_2} + \overline{z_1}z_2), [z_1, z_2] = \frac{1}{2i}(z_1\overline{z_2} - \overline{z_1}z_2).$

Proposition 4. *If* $z_1 = r_1(\cos\alpha_1 + i\sin\alpha_1), z_2 = r_2(\cos\alpha_2 + i\sin\alpha_2)$ *where* $r_1, r_2 \square 0$ *then*

- $|z_1z_2| = |z_1||z_2|$, $|\frac{z_1}{z_2}| = \frac{|z_1|}{|z_2|}$ and $|z_1 + z_2| \square |z_1| + |z_2|$.
- $z_1z_2 = r_1r_2[\cos(\alpha_1+\alpha_2) + i\sin(\alpha_1+\alpha_2)]$
- $\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\alpha_1-\alpha_2) + i\sin(\alpha_1-\alpha_2)]$ where $r_2 > 0$.
- $< z_1, z_2 > = |z_1||z_2|\cos(\alpha_1-\alpha_2)$ and $< z_1, z_2 > = < z_2, z_1 >$.
- $< az_1 + bz_3, z_2 > = a < z_1, z_2 > + b < z_3, z_2 >$ forall complex numbers z_1, z_2, z_3 and forall $a, b \in R$.
- $[z_1, z_2] = |z_1||z_2|\sin(\alpha_2-\alpha_1)$ and $[z_1, z_2] = -[z_2, z_1]$.
- Given $z_1 = \cos\alpha_1 + i\sin\alpha_1, z_2 = \cos\alpha_2 + i\sin\alpha_2$ we have

$$z_1 - z_2 = i2\sin\frac{\alpha_1-\alpha_2}{2}(\cos\frac{\alpha_1+\alpha_2}{2} + i\sin\frac{\alpha_1+\alpha_2}{2})$$

$$|z_1 - z_2| = 2|\sin\frac{\alpha_1-\alpha_2}{2}|.$$

Proof: It's easier to get the result.

You can check all the results. Special, deviation product hasgeometric significance: Given point O and the coordinates of the points M, N are respectively z_1, z_2, three points O, M, N all on a line if and only if $[z_1, z_2] = 0$. When three points O, M, N not in line then absolute value of deviation product $[z_1, z_2]$ twice the area triangle OMN.

Given two complex numbers z_1 and z_2 we always have

$$z_1 = z_2 \Leftrightarrow |z_1| = |z_2|, \arg z_1 = \arg z_2 + 2k\pi, k \in Z.$$

$$\arg(z_1z_2) = \arg(z_1) + \arg(z_2) + 2k\pi, k \in Z.$$

$$\arg(\frac{z_1}{z_2}) = \arg(z_1) - \arg(z_2) + 2k\pi, k \in Z.$$

$$Arg(z_1z_2) = Arg(z_1) + Arg(z_2).$$

$$Arg(\frac{z_1}{z_2}) = Arg(z_1) - Arg(z_2).$$

We say that, the polynomials of positive degree in $C[x]$ always have roots in C. The fundamental theorem of algebra states that every non-constant single-variable polynomial with complex coefficients has at least one complex root. This includes polynomials with real coefficients, since every real number is a complex number with an imaginary part equal to zero.

Equivalently (by definition), the theorem states that the field of complex numbers is algebraically closed.

Definition 2. *In abstract algebra, an algebraically closed field* K *contains a root for every non-constant polynomial in* $K[x]$, *the ring of polynomials in the variable* x *with coefficients in* K.

From The fundamental theorem of algebra, we deduce results of irreducible polynomial in $C[x]$:

Corollary 4. *Given the polynomials in* $C[x]$ *where degree* $n > 0$ *have* n *roots in* C *and the irreducible polynomials in* $C[x]$ *have 1 degree.*

Corollary 5. *Let* $f(x) \in R[x] \setminus R$. $f(x)$ *is irreducible polynomial if and only if or* $f(x) = ax + b$ *where* $a \neq 0$ *or* $f(x) = ax^2 + bx + c$ *where* $a \neq 0$ *and* $b^2 - 4ac < 0$.

2. SOME EXAMPLES OF COMPLEX NUMBERS

Example 2. *Prove that, with two complex numbers* z_1 *and* z_2 *we always have*

$$2|z_1|^2 + 2|z_2|^2 = |z_1 + z_2|^2 + |z_1 - z_2|^2.$$

Proof: Assume $A(z_1), B(z_2)$ and $C(z_1 + z_2)$. So the quadrangle $OACB$ is Parallelogram, we deduce $OC^2 + AB^2 = 2OA^2 + 2OB^2$ or $2[|z_1|^2 + |z_2|^2] = |z_1 - z_2|^2 + |z_1 + z_2|^2$.

Example 3. *With two complex numbers* z *and* z', *we set* $u = \sqrt{zz'}$. *Prove that*

$$|z| + |z'| = |\frac{z+z'}{2} - u| + |\frac{z+z'}{2} + u|.$$

Proof: Because

$$|\frac{z+z'}{2} - u| + |\frac{z+z'}{2} + u| = |\frac{(\sqrt{z} - \sqrt{z'})^2}{2}| + |\frac{(\sqrt{z} + \sqrt{z'})^2}{2}|$$

if we set $z_1 = \sqrt{z}, z_2 = \sqrt{z'}$ then we will have to prove: $2[|z_1|^2 + |z_2|^2] = |z_1 - z_2|^2 + |z_1 + z_2|^2$.

We consider $A(z_1), B(z_2)$ and $C(z_1 + z_2)$. Because the quadrangle $OACB$ is Parallelogram, deduce $OC^2 + AB^2 = 2OA^2 + 2OB^2$ or $2[|z_1|^2 + |z_2|^2] = |z_1 - z_2|^2 + |z_1 + z_2|^2$.

Example 4. [N+15] *Prove that, with three different complex numbers* z_1, z_2, z_3 *have*

$$2\left|\frac{z_1+z_2}{2}-\sqrt{z_1z_2}\right|+2\left|\frac{z_1+z_3}{2}-\sqrt{z_1z_3}\right|$$
$$=\left|\frac{z_2+z_3}{2}-\sqrt{z_2z_3}\right|+2\left|z_1+\frac{z_2+z_3}{4}-\sqrt{z_1z_2}-\sqrt{z_1z_3}+\frac{\sqrt{z_2z_3}}{2}\right|.$$

Proof: Set $u_j=\sqrt{z_j}$ where $j=1,2,3$. We will have to prove

$$|u_1-u_2|^2+|u_1-u_3|^2=\frac{|u_2-u_3|^2}{2}+2\left|u_1-\frac{u_2+u_3}{2}\right|^2.$$

Consider triangle ABC where $A(u_1), B(u_2), C(u_3)$ and $M(\frac{u_2+u_3}{2})$ is midpoint of sided BC.

Because $b^2+c^2=2m_a^2+\frac{a^2}{2}$ deduce we get

$$|u_1-u_2|^2+|u_1-u_3|^2=\frac{|u_2-u_3|^2}{2}+2\left|u_1-\frac{u_2+u_3}{2}\right|^2$$

and hence, ends the proof.

Example 5. [Euler] *Given* $x_1,x_2,x_3,x_4,y_1,y_2,y_3,y_4\in R$ *we always have* $(x_1^2+x_2^2+x_3^2+x_4^2)(y_1^2+y_2^2+y_3^2+y_4^2)=u_1^2+u_2^2+u_3^2+u_4^2$, where

$$\begin{cases} u_1=x_1y_1-x_2y_2-x_3y_3-x_4y_4 \\ u_2=x_1y_2+x_2y_1+x_3y_4-x_4y_3 \\ u_3=x_1y_3-x_2y_4+x_3y_1+x_4y_2 \\ u_4=x_1y_4+x_2y_3-x_3y_2+x_4y_1 \end{cases}$$

and deduce inequality $u_1^2+u_2^2+u_3^2+u_4^2 \square (x_1y_1+x_2y_2+x_3y_3+x_4y_4)^2$.

Proof: Set $z_1=x_1+ix_2, z_2=x_3+ix_4, z_3=y_1+iy_2$ and $z_4=y_3+iy_4$. We have

$$T=(x_1^2+x_2^2+x_3^2+x_4^2)(y_1^2+y_2^2+y_3^2+y_4^2)$$
$$=(z_1\overline{z_1}+z_2\overline{z_2})(z_3\overline{z_3}+z_4\overline{z_4})=z_1z_3\overline{z_1z_3}+z_2z_3\overline{z_2z_3}+z_1z_4\overline{z_1z_4}+z_2z_4\overline{z_2z_4}.$$

Therefore $T=u_1^2+u_2^2+u_3^2+u_4^2$.

Example 6. *Given three different complex numbers* z_1,z_2,z_3 *we have*

$$f(x)=\frac{z_1^n(x-z_2)(x-z_3)}{(z_1-z_2)(z_1-z_3)}+\frac{z_2^n(x-z_3)(x-z_1)}{(z_2-z_3)(z_2-z_1)}+\frac{z_3^n(x-z_1)(x-z_2)}{(z_3-z_1)(z_3-z_2)}=x^n$$

với $n=0,1,2$.

Proof: The polynomial $f(x)-x^n$ and its degree is less than 3 such that $f(z_1)=f(z_2)=f(z_3)=0$. Therefore $f(x)=x^n$ and we get the result.

Example 7. *Given three different complex numbers* z_1,z_2,z_3 *and two complex numbers* u,v *we have*

$$\frac{(x-u)(x-v)}{(x-z_1)(x-z_2)(x-z_3)}=\frac{(z_1-u)(z_1-v)}{(z_1-z_2)(z_1-z_3)(x-z_1)}+\frac{(z_2-u)(z_2-v)}{(z_2-z_1)(z_2-z_3)(x-z_2)}+\frac{(z_3-u)(z_3-v)}{(z_3-z_1)(z_3-z_2)(x-z_3)}.$$

Proof: We consider

$$\frac{(x-u)(x-v)}{(x-z_1)(x-z_2)(x-z_3)}=\frac{x_1}{x-z_1}+\frac{x_2}{x-z_2}+\frac{x_3}{x-z_3}.$$

Then we have $(x-u)(x-v)=x_1(x-z_2)(x-z_3)+x_2(x-z_3)(x-z_1)+x_3(x-z_1)(x-z_2)$.

Where $x=z_1,z_2,z_3$ we get

$$\begin{cases} x_1=\dfrac{(z_1-u)(z_1-v)}{(z_1-z_2)(z_1-z_3)} \\ x_2=\dfrac{(z_2-u)(z_2-v)}{(z_2-z_3)(z_2-z_1)} \\ x_3=\dfrac{(z_3-u)(z_3-v)}{(z_3-z_1)(z_3-z_2)} \end{cases}$$

and we get the result.

Example 8. *Given three different complex numbers* z_1,z_2,z_3 *we have*

$$\frac{z_2+z_3}{(z_1-z_2)(z_1-z_3)(z_1-t)}+\frac{z_3+z_1}{(z_2-z_3)(z_2-z_1)(z_2-t)}+\frac{z_1+z_2}{(z_3-z_1)(z_3-z_2)(z_3-t)}=\frac{t-z_1-z_2-z_3}{(t-z_1)(t-z_2)(t-z_3)}.$$

Proof: We consider

$$\frac{t-z_1-z_2-z_3}{(t-z_1)(t-z_2)(t-z_3)}=\frac{x}{t-z_1}+\frac{y}{t-z_2}+\frac{z}{t-z_3}.$$

then we have $t-z_1-z_2-z_3=x(t-z_2)(t-z_3)+y(t-z_3)(t-z_1)+z(t-z_1)(t-z_2)$.

where $t=z_1,z_2,z_3$ we get

$$\begin{cases} x=-\dfrac{z_2+z_3}{(z_1-z_2)(z_1-z_3)} \\ y=-\dfrac{z_3+z_1}{(z_2-z_3)(z_2-z_1)} \\ z=-\dfrac{z_1+z_2}{(z_3-z_1)(z_3-z_2)} \end{cases}$$

and we get the result.

Example 9. *Given four different complex numbers* z_1,z_2,z_3,z_4 *and three complex numbers* u,v,w *then we have* $\dfrac{(x-u)(x-v)(x-w)}{(x-z_1)(x-z_2)(x-z_3)(x-z_4)}=$

$$\frac{(z_1-u)(z_1-v)(z_1-w)}{(z_1-z_2)(z_1-z_3)(z_1-z_4)(x-z_1)}+\frac{(z_2-u)(z_2-v)(z_2-w)}{(z_2-z_1)(z_2-z_3)(z_2-z_4)(x-z_2)}$$

$$+\frac{(z_3-u)(z_3-v)(z_3-w)}{(z_3-z_1)(z_3-z_2)(z_3-z_4)(x-z_3)}+\frac{(z_4-u)(z_4-v)(z_4-w)}{(z_4-z_1)(z_4-z_2)(z_4-z_3)(x-z_4)}.$$

Proof: We consider

$$\frac{(x-u)(x-v)(x-w)}{(x-z_1)(x-z_2)(x-z_3)(x-z_4)}=\frac{x_1}{x-z_1}+\frac{x_2}{x-z_2}+\frac{x_3}{x-z_3}+\frac{x_4}{x-z_4}.$$

Then we have

$$(x-u)(x-v)(x-w)=x_1(x-z_2)(x-z_3)(x-z_4)+x_2(x-z_3)(x-z_4)(x-z_1)+x_3(x-z_4)(x-z_1)(x-z_2)$$

$+x_4(x-z_1)(x-z_2)(x-z_3).$ Where

$x=z_1,z_2,z_3,z_4$ have

$$\begin{cases} x_1=\dfrac{(z_1-u)(z_1-v)(z_1-w)}{(z_1-z_2)(z_1-z_3)(z_1-z_4)} \\ x_2=\dfrac{(z_2-u)(z_2-v)(z_2-w)}{(z_2-z_3)(z_2-z_4)(z_2-z_1)} \\ x_3=\dfrac{(z_3-u)(z_3-v)(z_3-w)}{(z_3-z_4)(z_3-z_1)(z_3-z_2)} \\ x_4=\dfrac{(z_4-u)(z_4-v)(z_4-w)}{(z_4-z_1)(z_4-z_2)(z_4-z_3)} \end{cases}$$

and we get the result.

Example 10. *Given three different complex numbers* z_1,z_2,z_3 *we have*

$$\frac{(z_1+z_2)(z_2+z_3)}{(z_1-z_2)(z_2-z_3)}+\frac{(z_2+z_3)(z_3+z_1)}{(z_2-z_3)(z_3-z_1)}+\frac{(z_3+z_1)(z_1+z_2)}{(z_3-z_1)(z_3-z_2)}=-1.$$

Proof: Set $x=\dfrac{z_1+z_2}{z_1-z_2},\quad y=\dfrac{z_2+z_3}{z_2-z_3}, z=\dfrac{z_3+z_1}{z_3-z_1}.$

Then we have

$$(x+1)(y+1)(z+1)=(x-1)(y-1)(z-1).$$

Therefore we have $xy+yz+zx=-1$ and we get the result.

Example 11. *Given three different complex numbers* z_1,z_2,z_3 *we have*

$$\frac{z_1^2}{(z_1-z_2)(z_1-z_3)(z-z_1)}+\frac{z_2^2}{(z_2-z_3)(z_2-z_1)(z-z_2)}$$

$$+\frac{z_3^2}{(z_3-z_1)(z_3-z_2)(z-z_3)}=\frac{z^2}{(z-z_1)(z-z_2)(z-z_3)}.$$

Proof: Set

$$\frac{z^2}{(z-z_1)(z-z_2)(z-z_3)}=\frac{x_1}{z-z_1}+\frac{x_2}{z-z_2}+\frac{x_3}{z-z_3}.$$

Then we have

$$z^2=x_1(z-z_2)(z-z_3)+x_2(z-z_3)(z-z_1)+x_3(z-z_1)(z-z_2).$$

Where $z=z_1,z_2,z_3$

we get
$$\begin{cases} x_1=\dfrac{z_1^2}{(z_1-z_2)(z_1-z_3)} \\ x_2=\dfrac{z_2^2}{(z_2-z_3)(z_2-z_1)} \\ x_3=\dfrac{z_3^2}{(z_3-z_1)(z_3-z_2)} \end{cases}$$

and we deduce the result.

Example 12. *Given* $n+1$ *complex numbers* $z_1,z_2,\ldots,z_n,z$ *and* $z\neq z_k, k=1,2,\ldots,n,$ *we have*

$$\frac{z_1-z_n}{(z-z_1)(z-z_n)}=\frac{z_1-z_2}{(z-z_1)(z-z_2)}+\frac{z_2-z_3}{(z-z_2)(z-z_3)}+\cdots+\frac{z_{n-1}-z_n}{(z-z_{n-1})(z-z_n)}.$$

Proof: From $\dfrac{z_k-z_h}{(z-z_k)(z-z_h)}=\dfrac{1}{z-z_k}-\dfrac{1}{z-z_h}$ we deduce the result.

Example 13. *Assume the different complex numbers* $z_1,z_2,\ldots,z_s$ *and* $z_i+a_j\neq 0$ *where* $i=1,\ldots,s$ *and* $j=1,2,\ldots,n.$

Given $x_1,x_2,\ldots,x_n$ *by*

$$\frac{(z-z_1)(z-z_2)\ldots(z-z_s)}{(z+a_1)(z+a_2)\ldots(z+a_n)}=\frac{x_1}{z+a_1}+\frac{x_2}{z+a_2}+\cdots+\frac{x_n}{z+a_n}$$

and deduce

$$\sum_{k=1}^{n}\frac{\prod_{i=1}^{s}|a_k+z_i|}{\prod_{i=1}^{k-1}|a_k-a_i|\prod_{i=k+1}^{n}|a_i-a_k\|z+a_k|}\square\frac{|z-z_1|\ldots|z-z_s|}{|z+a_1|\ldots|z+a_n|}.$$

Proof: Assume $\dfrac{(z-z_1)\ldots(z-z_s)}{(z+a_1)\ldots(z+a_n)}=\dfrac{x_1}{z+a_1}+\dfrac{x_2}{z+a_2}+\cdots+\dfrac{x_n}{z+a_n}.$

then we have

$$x_1(z+a_2)(z+a_3)\ldots(z+a_n)+x_2(z+a_1)(z+a_3)\ldots(z+a_n)$$
$$+x_3(z+a_1)(z+a_2)\ldots(z+a_n)+x_4(z+a_1)(z+a_2)\ldots(z+a_n)$$
$$+\ldots+x_n(z+a_1)(z+a_2)\ldots(z+a_{n-1})$$
$$=(z-z_1)(z-z_2)\ldots(z-z_s).$$

we get

$$\begin{cases} x_1 = \dfrac{(-1)^s \prod_{i=1}^{s}(a_1 + z_i)}{\prod_{i=2}^{n}(a_i - a_1)} \text{ where } x = -a_1 \\ x_2 = \dfrac{(-1)^{s+1} \prod_{i=1}^{s}(a_2 + z_i)}{(a_2 - a_1)\prod_{i=3}^{n}(a_i - a_2)} \text{ where } x = -a_2 \\ x_3 = \dfrac{(-1)^{s+2} \prod_{i=1}^{s}(a_3 + z_i)}{\prod_{i=1}^{2}(a_3 - a_i)\prod_{i=4}^{n}(a_i - a_3)} \text{ where } x = -a_3 \\ \quad\quad \ldots \\ x_n = \dfrac{(-1)^{s+n-1} \prod_{i=1}^{s}(a_n + z_i)}{\prod_{i=1}^{n-1}(a_n - a_i)} \text{ where } x = -a_n \end{cases}$$

and

$$\frac{(z-z_1)\ldots(z-z_s)}{(z+a_1)\ldots(z+a_n)} = \sum_{k=1}^{n} \frac{(-1)^{s+k-1}\prod_{i=1}^{s}(a_k + z_i)}{\prod_{i=1}^{k-1}(a_k - a_i)\prod_{i=k+1}^{n}(a_i - a_k)(z + a_k)}.$$

Therefore, we deduce

$$\sum_{k=1}^{n} \frac{\prod_{i=1}^{s}|a_k + z_i|}{\prod_{i=1}^{k-1}|a_k - a_i|\prod_{i=k+1}^{n}|a_i - a_k||z + a_k|} \square \frac{|z-z_1|\ldots|z-z_s|}{|z+a_1|\ldots|z+a_n|}.$$

3. CONCLUSION

Many teachers believe that they in Vietnam have benefited from innovative general education program after 2015. Teaching method had important innovation in both theory and practical. it is important to teach pupils self-learning and discovering knowledge. teachers help pupils to applying knowledge flexibly and studying more effectively. I and my partner ([HT15, Tin15]) state that, It has been my experience that competency in mathematics, both in complex numbers manipulations and in understanding its conceptual foundations, Enhances a teacher's ability to teach mathematical in high schools in Vietnam.

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