# DERIVATIONS FOR THE FAMILIES OF GENERALIZED DISTRIBUTIONS

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***ABSTRACT:*** *Generalization of distributions is usually motivated by limitations in characteristics of existing distributions so as to introduce more flexibility and improve goodness of fit. This is done by parameter induction into an existing distribution and therefore remains an approach to generalizing distributions. In this article, families of generalized distributions are generated by sequential application of methods in permutations of five distinct parameter induction methods: Lehmann Alternative 1 (LA1); Lehmann Alternative 2 (LA2); Marshal and Olkin Method (M-OM); α-Power Transformation (APT); and Power Transformation Method (PTM). This is done by taken two methods at a time. Sixteen distinct families of generalized distributions were generated. Some of the families of generalized distributions obtained are already in existence while several others are entirely new.*
***KEYWORDS:*** *Generalized distributions; Lehmann Alternatives; α-Power Transformation Method; Power Transformation Method; Marshall and Olkin Method.*

## 1. INTRODUCTION

There are several parameter induction methods in literature. These methods have received many applications and produced more flexible distributions that improved goodness of fit. ([MO97]) introduced a method for adding a parameter to a base distribution. The Probability Density Function (PDF) and Cumulative Distribution Function (CDF) of any random variable belonging to the family of generalized distributions obtained from this method for $x \in \Re$ and any $\theta \in (0, \infty)$ are

$$f(x;\theta) = \frac{\theta g(x)}{(1-(1-\theta)(1-G(x)))^2} \tag{1}$$

and

$$F(x;\theta) = \frac{G(x)}{1-(1-\theta)(1-G(x))} \tag{2}$$

$\theta$ is the introduced parameter, where g(x) and G(x) are the PDF and CDF of any base distribution, ([MK16]) proposed another method of introducing an extra parameter (α>0) to a family of distributions in what they called α-Power Transformation Method. The α-Power transformations of the CDF (F(x)) and PDF (f(x)) of a continuous variable X for $x \in \Re$ are defined as follows:

$$F_{APT}(x) = \alpha^{F(x)} - 1 \Big/ \alpha - 1 \text{ if } \alpha \neq 1 \tag{3}$$

and

$$f_{APT}(x) = \frac{\log \alpha}{\alpha - 1} f(x)\alpha^{F(x)} \quad \text{if } \alpha \neq 1 \tag{4}$$

When α=1, The PDF and CDF becomes those of the base distributions.

([TN15]) discussed Lehmann Alternatives as ways of obtaining Exponentiated families of distributions with additional parameter (a>0). For any baseline distribution having PDF (g(z)) and CDF (G(z)), Lehmann Alternative 1 has the following PDF and CDF,

$$f(z) = ag(z)G(z)^{a} \tag{5}$$

and

$$F(z) = G(z)^{a} \tag{6}$$

Whereas the PDF and CDF obtained using Lehmann Alternative 2 are

$$f(z) = ag(z)[1\text{-}G(z)]^{a}$$

and

$$F(z) = 1\text{-}[1\text{-}G(z)]^{a} \tag{7}$$

([GK09]) gave different interpretations to these exponentiated families of distributions and also discussed the Power Transformation method of parameter induction. Suppose X is random variable, then for an additional parameter (α > 0), consider a new random variable Y such that $Y = X^{1/\alpha}$, then the corresponding function PDF and CDF satisfy the following;

$$f(y) = \alpha x^{\alpha\text{-}1} f(x^{\alpha})$$

and

$$F(y) = F(x^{\alpha}) \tag{8}$$

Other parameter induction methods and applications can be found in ([GK09]), ([LFA13]), and ([TN15]).
This article generates families of generalized distributions with two additional parameters. Methods can be used similarly to obtain generalized distributions introducing more than two parameters. In the next section, we obtain and present some functions of derived families of generalized distributions while conclusion is done in section 3.

## 2. FAMILIES OF GENERALIZED DISTRIBUTIONS AND THEIR FUNCTIONS

Let X be a base random variable having the following functions;
Probability Density Function denoted by $f(x)$, Cumulative Distribution Function denoted by $F(x)$. If X is a lifetime random variable, then, the Survival Function (SF) is denoted by, the Hazard Function (HF) is denoted by $h(x)$, and the Reversed Hazard Function (RHF) is denoted by $r(x)$.
Let Y be a continuous random variable obtained by introducing a parameter (c>0) to the base random variable X, and let Z be another continuous variable belonging to a family of generalized distribution obtained by introducing another parameter(t>0) to Y.
Then to generate families of generalized distributions introducing two parameters, we first obtain permutations of five distinct parameter induction methods (those reviewed) taken two methods at a time. The methods in each permutation are then applied sequentially to obtain a family of generalized distributions. Below are the families obtained and some of their statistical functions.
**Permutation 1**: LA1, LA2
Applying LA1 first and then LA2,
**Family 1**:

$$f_Z(x) = ct(F(x))^{c-1} f(x)(1-F(x)^{c})^{t-1}$$

$$
\begin{aligned}
&F_Z(x) = 1-(1-(F(x))^c)^t, \\
&\overline{F}_Z(x) = (1-(F(x))^c)^t \\
&h_Z(x) = ct(F(x))^{c-1} f(x)(1-(F(x))^c)^{-1} \\
&r_Z(x) = \frac{ct(F(x))^{c-1} f(x)(1-(F(x))^c)^{t-1}}{1-(1-(F(x))^c)^t}
\end{aligned}
\tag{9}
$$

**Permutation 2**: LA2, LA1
Applying LA2 first and then LA1;
**Family 2**:

$$
\begin{aligned}
&f_Z(x) = ct(\overline{F}(x))^{c-1} f(x)(1-(\overline{F}(x))^c)^{t-1} \\
&F_Z(x) = (1-(\overline{F}(x))^c)^t, \\
&\overline{F}_Z(x) = 1-(1-(\overline{F}(x))^c)^t \\
&h_Z(x) = \frac{ct(\overline{F}(x))^{c-1} f(x)(1-(\overline{F}(x))^c)^{t-1}}{1-(1-(\overline{F}(x))^c)^t} \\
&r_Z(x) = ct(\overline{F}(x))^{c-1} f(x)(1-(\overline{F}(x))^c)^{-1}
\end{aligned}
\tag{10}
$$

**Permutation 3**: LA2, PTM
Apply LA2 first and then PTM
**Family 3**:

$$
\begin{aligned}
&f_Z(x) = ctx^{t-1} f(x^t)(\overline{F}(x^t))^{c-1} \\
&F_Z(x) = 1-(\overline{F}(x^t))^c, \ \overline{F}_Z(x) = (\overline{F}(x^t))^c \\
&h_Z(x) = ctx^{t-1} f(x^t)(\overline{F}(x^t))^{-1} \\
&r_Z(y) = ctx^{t-1} f(x^t)(\overline{F}(x^t))^{c-1}\left(1-(\overline{F}(x^t))^c\right)^{-1}
\end{aligned}
\tag{11}
$$

**Permutation 4**: PTM, LA2
Apply PTM first and then LA2
**Family 4**:

$$
\begin{aligned}
&f_Z(x) = ctx^{c-1} f(x^c)(\overline{F}(x^c))^{t-1} \\
&F_Z(x) = 1-(\overline{F}(x^c))^t, \ \overline{F}_Z(x) = (\overline{F}(x^c))^t \\
&h_Z(x) = ctx^{c-1} f(x^c)(1-F(x^c))^{-1} \\
&r_Z(x) = ctx^{c-1} f(x^c)(1-F(x^c))^{t-1}\left(1-(1-F(x^c))^t\right)^{-1}
\end{aligned}
\tag{12}
$$

From equations in (11) and those in (12) it is observed that PTM and LA2 are commutative.
**Permutation 5**: LA2, APT
Apply LA2 first and then APT
**Family 5**:

$$
\begin{aligned}
f_z(x) &= \begin{cases} \dfrac{\log t}{t-1} c[\bar{F}(x)]^{c-1} f(x) t^{1-[\bar{F}(x)]^c} \\ c[\bar{F}(x)]^{c-1} f(x), when\,(t=1) \end{cases} \\
\bar{F}_z(x) &= \begin{cases} t - t^{1-[\bar{F}(x)]^c} \big/ t-1 \\ [\bar{F}(x)]^c, when\,(t=1) \end{cases} \\
F_z(x) &= \begin{cases} t^{1-[\bar{F}(x)]^c} - 1 \big/ t-1 \\ 1-[\bar{F}(x)]^c, when\,(t=1) \end{cases} \\
h_z(x) &= \begin{cases} c\log(t)[\bar{F}(x)]^{c-1} f(x) t^{1-[\bar{F}(x)]^c} \big/ t - t^{1-[\bar{F}(x)]^c} \\ cf(x)[\bar{F}(x)]^{-1}, when\,(t=1) \end{cases} \\
r_z(x) &= \begin{cases} c\log(t)[\bar{F}(x)]^{c-1} f(x) t^{1-[\bar{F}(x)]^c} \big/ t^{1-[\bar{F}(x)]^c} - 1 \\ cf(x)[\bar{F}(x)]^{c-1}[1-(\bar{F}(x))^c]^{-1}, when\,(t=1) \end{cases}
\end{aligned}
\tag{13}
$$

**Permutation 6**: APT, LA2
Apply APT first and then LA2
**Family 6**:

$$
\begin{aligned}
f_z(x) &= \begin{cases} t\left(c - c^{F(x)} \big/ c-1\right)^{t-1} (\log c / c-1) f(x) c^{F(x)} \\ t[\bar{F}(x)]^{t-1} f(x), when\,(c=1) \end{cases} \\
F_z(x) &= \begin{cases} 1-(c - c^{F(x)} \big/ c-1)^t \\ 1-[\bar{F}(x)]^{t}, when\,(c \neq 1) \end{cases} \\
\bar{F}_z(x) &= \begin{cases} (c - c^{F(x)} \big/ c-1)^t \\ [\bar{F}(x)]^t, when\,(t=1) \end{cases} \\
h_z(x) &= \begin{cases} t(\log c / c-1) f(x) c^{F(x)} (c - c^{F(x)} \big/ c-1)^{-1} \\ tf(x)[\bar{F}(x)]^{-1}, when\,(c=1) \end{cases} \\
r_z(x) &= \begin{cases} t(\log c / c-1) f(x) c^{F(x)} (c - c^{F(x)} \big/ c-1)^{t-1} \\ tf(x)[\bar{F}(x)]^{t-1}[1-[(\bar{F}(x))^c]^t]^{-1}, when\,(c=1) \end{cases}
\end{aligned}
\tag{14}
$$

**Permutation 7**: M-OM, LA2
Apply M-OM first and then LA2
**Family 7**:

$$
\begin{aligned}
f_z(x) &= \frac{c^t t(\bar{F}(x))^{t-1} f(x)}{(1-(1-c)(1-F(x)))^{t+1}} \\
F_z(x) &= 1-\left(\frac{c\bar{F}(x)}{1-(1-c)(1-F(x))}\right)^t \\
\bar{F}_z(x) &= \left(\frac{c\bar{F}(x)}{1-(1-c)(1-F(x))}\right)^t \\
h_z(x) &= \frac{tf(x)}{(1-(1-c)(1-F(x)))\bar{F}(x)} = \frac{th(x)}{1-(1-c)(1-F(x))}
\end{aligned}
\tag{15}
$$

$$r_z(x)=\frac{c^t t(\bar{F}(x))^{t-1} f(x)}{(1-(1-c)(1-F(x)))\left((1-(1-c)(1-F(x)))^t-(t\bar{F}(x))^t\right)}$$

**Permutation 8**: LA2, M-OM
Apply LA2 first and then M-OM
**Family 8**:

$$\begin{aligned}
f_z(x)&=\frac{ct(\bar{F}(x))^{c-1}f(x)}{\left(1-(\bar{F}(x))^c+t(\bar{F}(x))^c\right)^2}=\frac{ct(\bar{F}(x))^{c-1}f(x)}{\left(1-(1-t)(\bar{F}(x))^c\right)^2}\\
F_z(x)&=\frac{1-(\bar{F}(x))^c}{1-(\bar{F}(x))^c+t(\bar{F}(x))^c}=\frac{1-(\bar{F}(x))^c}{1-(1-t)(\bar{F}(x))^c}\\
\bar{F}_z(x)&=\frac{t(\bar{F}(x))^c}{1-(\bar{F}(x))^c+t(\bar{F}(x))^c}=\frac{t(\bar{F}(x))^c}{1-(1-t)(\bar{F}(x))^c}\\
h_z(x)&=\frac{ct(\bar{F}(x))^{-1}f(x)}{1-(\bar{F}(x))^c+t(\bar{F}(x))^c}=\frac{ct(\bar{F}(x))^{-1}f(x)}{1-(1-t)(\bar{F}(x))^c}\\
r_z(x)&=\frac{ct(\bar{F}(x))^{c-1}f(x)\left(1-(\bar{F}(x))^c\right)^{-1}}{1-(\bar{F}(x))^c+t(\bar{F}(x))^c}
\end{aligned}\tag{16}$$

**Permutation 9**: LA1, PTM
Apply LA1 first and then PTM
**Family 9**:

$$\begin{aligned}
f_z(x)&=ctx^{t-1}f(x^t)(F(x^t))^{c-1}\\
F_z(x)&=(F(x^t))^c,\\
\bar{F}_z(x)&=1-(F(x^t))^c\\
h_z(x)&=ctx^{t-1}f(x^t)(F(x^t))^{c-1}\left(1-(F(x^t))^c\right)^{-1}\\
r_z(x)&=ctx^{t-1}f(x^t)(F(x^t))^{-1}=ctx^{t-1}r(x^t)
\end{aligned}\tag{17}$$

**Permutation 10**: PTM, LA1
Apply PTM first and then LA1
**Family 10**:

$$\begin{aligned}
f_z(x)&=ctx^{c-1}f(x^c)(F(x^c))^{t-1}\\
F_z(x)&=(F(x^c))^t,\\
\bar{F}_z(x)&=1-(F(x^c))^t\\
h_z(x)&=ctx^{c-1}f(x^c)(F(x^c))^{t-1}\left(1-(F(x^c))^t\right)^{-1}\\
r_z(x)&=ctx^{c-1}f(x^c)(F(x^c))^{-1}=ctx^{c-1}r(x^c)
\end{aligned}\tag{18}$$

From equations in (17) and those in (18), it is observed that LA1 and PTM are commutative.

**Permutation 11**: LA1, APT
Apply LA1 first and then APT
**Family 11**:

$$
\begin{aligned}
f_z(x) &= \begin{cases} c(\log t \big/ t-1)\, f(x)[F(x)]^{c-1} t^{[F(x)]^c} \\ cf(x)[F(x)]^{c-1}, \, when\,(t=1) \end{cases} \\
F_z(x) &= \begin{cases} t^{[F(x)]^c} - 1 \big/ t-1 \\ [F(x)]^c, \, when\,(t=1) \end{cases} \\
\bar{F}_z(x) &= \begin{cases} t - t^{[F(x)]^c} \big/ t-1 \\ [1-(F(x))^c]^t, \, when\,(t=1) \end{cases} \\
h_z(x) &= \begin{cases} c\log(t)\, f(x)[F(x)]^{c-1} t^{[F(x)]^c} \big/ t - t^{[F(x)]^c} \\ cf(x)[F(x)]^{c-1}[1-(F(x))^c]^{-1}, \, when\,(t=1) \end{cases} \\
r_z(x) &= \begin{cases} c\log(t)\, f(x)[F(x)]^{c-1} t^{[F(x)]^c} \big/ t^{[F(x)]^c} - 1 \\ cf(x)[F(x)]^{-1}, \, when\,(t=1) \end{cases}
\end{aligned} \tag{19}
$$

**Permutation 12**: APT, LA1
Apply APT first and then LA1
**Family 12**:

$$
\begin{aligned}
f_z(x) &= \begin{cases} t(\log c \big/ c-1)\, f(x) c^{F(x)} (c^{F(x)} - 1 \big/ c-1)^{t-1} \\ tf(x)[F(x)]^{t-1}, \, when\,(c=1) \end{cases} \\
F_z(x) &= \begin{cases} (c^{F(x)} - 1 \big/ c-1)^t \\ [F(x)]^t, \, when\,(c=1) \end{cases} \\
\bar{F}_z(x) &= \begin{cases} 1-(c^{F(x)} - 1 \big/ c-1)^t \\ 1-[F(x)]^t, \, when\,(c=1) \end{cases} \\
h_z(x) &= \begin{cases} t(\log c \big/ c-1)\, f(x) c^{F(x)} (c^{F(x)} - 1 \big/ c-1)^{t-1} \\ tf(x)[F(x)]^{t-1}[1-(F(x))^t]^{-1}, \, when\,(c=1) \end{cases} \\
r_z(x) &= \begin{cases} t(\log c \big/ c-1)\, f(x) c^{F(x)} [(c^{F(x)} - 1 \big/ c-1)]^{-1} \\ tf(x)[F(x)]^{-1}, \, when\,(c=1) \end{cases}
\end{aligned} \tag{20}
$$

**Permutation 13**: M-OM, LA1
Apply M-OM first and then LA1
**Family 13**:

$$
\begin{aligned}
f_z(x) &= \frac{ctf(x)(F(x))^{t-1}}{(1-(1-c)(1-F(x)))^{t+1}} \\
F_z(x) &= \left( \frac{F(x)}{1-(1-c)(1-F(x))} \right)^t
\end{aligned} \tag{21}
$$

$$\bar{F}_z(x) = 1 - \left( \frac{F(x)}{1-(1-c)(1-F(x))} \right)^t$$

$$h_z(x) = \frac{ctf(x)(F(x))^{t-1}}{(1-(1-c)(1-F(x)))((1-(1-c)(1-F(x)))^t - (F(x))^t)}$$

$$r_z(x) = \frac{ctf(x)}{F(x)(1-(1-c)(1-F(x)))} = \frac{ct \times r(x)}{1-(1-c)(1-F(x))}$$

**Permutation 14**: LA1, M-OM
Apply LA1 first and then M-OM
**Family 14**:

$$f_z(x) = \frac{ctf(x)(F(x))^{c-1}}{((F(x)^c + t(1-(F(x))^c))^2} = \frac{ctf(x)(F(x))^{c-1}}{(t+(1-t)(F(x))^c)^2}$$

$$F_z(x) = \frac{(F(x))^c}{(F(x))^c + t(1-(\bar{F}(x))^c)} = \frac{(F(x))^c}{t+(1-t)(F(x))^c}$$

$$\bar{F}_z(x) = \frac{t(1-(\bar{F}(x))^c)}{(F(x))^c + t(1-(\bar{F}(x))^c)} = \frac{t - t(\bar{F}(x))^c}{t+(1-t)(F(x))^c} \tag{22}$$

$$h_z(x) = \frac{c(F(x))^{c-1} f(x)}{(t+(1-t)(F(x))^c)(1-(F(x))^c)}$$

$$r_Z(x) = \frac{ctf(x)}{(t+(1-t)(F(x))^c)(F(x))}$$

**Permutation 15**: PTM, APT
Apply PTM first and then APT
**Family 15**:

$$f_z(x) = \begin{cases} c(\log t / t-1) x^{c-1} f(x^c) t^{F(x^c)} \\ cx^{c-1} f(x^c), when(t=1) \end{cases},$$

$$F_z(x) = \begin{cases} t^{F(x^c)} - 1 / t-1 \\ F(x^c), when(t=1) \end{cases}$$

$$\bar{F}_z(x) = \begin{cases} t - t^{F(x^c)} / t-1 \\ 1-F(x^c), when(t=1) \end{cases}$$

$$h_z(x) = \begin{cases} c\log(t) x^{c-1} f(x^c) t^{F(x^c)} / t - t^{F(x^c)} \\ cx^{c-1} h(x^c), when(t=1) \end{cases} \tag{23}$$

$$r_z(x) = \begin{cases} c\log(t) x^{c-1} f(x^c) t^{F(x^c)} / t^{F(x^c)} - 1 \\ cx^{c-1} r(x^c), when(t=1) \end{cases}$$

**Permutation 16**: APT, PTM
Apply APT first and then PTM
**Family 16**:

$$
\begin{aligned}
f_z(x) &= \begin{cases} tx^{t-1}(\log c / c-1) f(x^t) c^{F(x^t)} \\ tx^{t-1} f(x^t), when(c=1) \end{cases} \\
F_z(x) &= \begin{cases} c^{F(x^t)} - 1 / c-1 \\ F(x^t), when(c=1) \end{cases} \\
\bar{F}_Z(x) &= \begin{cases} 1-(c^{F(x^t)} - 1 / c-1) \\ 1-F(x^t), when(c=1) \end{cases} \\
h_z(x) &= \begin{cases} tx^{t-1} \log(c) f(x^t) c^{F(x^t)} / c - c^{F(x^t)} \\ tx^{t-1} h(x^t), when(c=1) \end{cases} \\
r_z(x) &= \begin{cases} tx^{t-1} \log(c) f(x^t) c^{F(x^t)} / c^{F(x^t)} - 1 \\ tx^{t-1} r(x^t), when(c=1) \end{cases}
\end{aligned} \tag{24}
$$

From equations in (23) and those in (25), it is observed that APT and PTM are commutative.
**Permutation 17**: M-OM, APT
Apply M-OM first and then APT
**Family 17**:

$$
\begin{aligned}
f_z(x) &= \begin{cases} (\log t / t-1)(cf(x) / (1-(1-c)(1-F(x)))^2) t^{(F(x)/(1-(1-c)(1-F(x)))} \\ cf(x) / (1-(1-c)(1-F(x)))^2, when(t=1) \end{cases} \\
F_z(x) &= \begin{cases} t^{(F(x)/(1-(1-c)(1-F(x)))} - 1 / t-1 \\ c\bar{F}(x)/1-(1-c)(1-F(x)), when(t=1) \end{cases} \\
\bar{F}_z(x) &= \begin{cases} (t - t^{(F(x)/1-(1-c)(1-F(x)))}) / t-1 \\ c\bar{F}(x)/1-(1-c)(1-F(x)), when(t=1) \end{cases} \\
h_z(x) &= \begin{cases} \dfrac{\log(t)(cf(x) / (1-(1-c)(1-F(x)))^2) t^{(F(x)/1-(1-c)(1-F(x)))}}{t - t^{(F(x)/1-(1-c)(1-F(x)))}} \\ h(x)/1-(1-c)(1-F(x), when(t=1) \end{cases} \\
r_z(x) &= \begin{cases} \dfrac{c\log(t)(cf(x) / (1-(1-c)(1-F(x)))^2) t^{(F(x)/1-(1-c)(1-F(x)))}}{t^{(F(x)/1-(1-c)(1-F(x)))} - 1} \\ cr(x)/1-(1-c)(1-F(x)), when(t=1) \end{cases}
\end{aligned} \tag{25}
$$

**Permutation 18**: APT, M-OM
Apply APT first and then M-OM
**Family 18**:

$$
f_z(x) = \begin{cases} \dfrac{t\log(c) f(x) c^{F(x)}}{(c-1)((c^{F(x)} - 1 / c-1) + t(c - c^{F(x)} / c-1))^2} \\ tf(x) / (1-(1-t)(1-F(x)))^2, when(c=1) \end{cases}
$$

$$F_z(x)=\begin{cases}(c^{F(x)}-1/c-1)(1/(c^{F(x)}-1/c-1)+t(c-c^{F(x)}/c-1))\\ F(x)/1-(1-t)(1-F(x)),\,when(c=1)\end{cases}$$

$$\bar{F}_z(x)=\begin{cases}t(c-c^{F(x)}/c-1)(1/(c^{F(x)}-1/c-1)+t(c-c^{F(x)}/c-1)\\ t\bar{F}(x)/1-(1-t)(1-F(x)),\,when(c=1)\end{cases}$$

$$h_z(x)=\begin{cases}\dfrac{\log(c)f(x)c^{F(x)}}{((c-c^{F(x)})((c^{F(x)}-1/c-1)+t(c-c^{F(x)}/c-1))}\\ h(x)/(1-(1-t)(1-F(x))),\,when(c=1)\end{cases}\quad(26)$$

$$r_z(x)=\begin{cases}\dfrac{t\log(c)f(x)c^{F(x)}}{(c^{F(x)}-1)(c^{F(x)}-1/c-1)+t(c-c^{F(x)}/c-1))}\\ tr(x)/(1-(1-t)(1-F(x))),\,when(c=1)\end{cases}$$

**Permutation 19**: M-OM, PTM
Apply M-OM first and then PTM
**Family 19**:

$$f_z(x)=\frac{ctx^{t-1}f(x^t)}{\left(1-(1-c)(1-F(x^t)\right)^2}$$

$$F_z(x)=\frac{F(x^t)}{1-(1-c)(1-F(x^t))}$$

$$\bar{F}_z(x)=\frac{t\bar{F}(x^t)}{1-(1-c)(1-F(x^t))}\quad(27)$$

$$h_z(x)=\frac{tx^{t-1}h(x^t)}{1-(1-c)(1-F(x^\beta))},$$

$$r_z(x)=\frac{ctx^{t-1}r(x^t)}{1-(1-c)(1-F(x^\beta))}$$

**Permutation 20**: PTM, M-OM
Apply PTM first and then M-OM
**Family 20**:

$$f_z(x)=\frac{ctx^{c-1}f(x^c)}{\left(1-(1-t)(1-F(x^c))\right)^2},$$

$$F_z(x)=\frac{F(x^c)}{1-(1-t)(1-F(x^c))}$$

$$\bar{F}_z(x)=\frac{t\bar{F}(x^c)}{1-(1-t)(1-F(x^c))},$$

$$h_z(x) = \frac{cx^{c-1}h(x^c)}{1-(1-t)(1-F(x^c))} \qquad (28)$$

$$r_z(x) = \frac{ctx^{c-1}r(x^c)}{1-(1-t)(1-F(x^c))}$$

From equations in (27) and (28), it is observed that PTM and M-OM are commutative. Methods 1 and 2 producing families 1 and 2 agree with those found in ([C+13]) and ([CD11])). Some generalizations of POM/MOE family of distributions can be found in ([JM08]) and ([TN15]).

## 3. CONCLUSIONS

Twenty methods producing sixteen distinct families of generalized distributions with two additional parameters were obtained. Models introducing two parameters in any base distribution may be generated from sequential applications of methods in permutations of $s \geq 2$ distinct parameter induction methods taken two methods at a time. In general, models introducing n parameters may be obtained by sequential application of methods in permutations of s ($s \geq n$) distinct parameter induction methods taken n methods at a time. Sequential application of methods in permutations of two distinct parameter induction methods (PTM inclusive) produced two generalized distributions belonging to the same family hence PTM and such other method are said to be commutative. Some of the families of generalized distributions generated are already in existence while several others are entirely new. Properties of new families of distribution need to be studied.

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