DYNAMIC MODELS FOR INTERNET NETWORKS DESCRIBED BY STOCHASTIC DELAY DIFFERENTIAL EQUATIONS

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ABSTRACT: In this paper, we consider Internet models, which respond to a congestion signal from the network described by a stochastic differential equation. We consider Internet networks with one source and n access links, as well as with r sources and single access link. We analyze the conditions for the existence of a solution and the algorithms needed to determine the solution. We carry out numerical simulations for certain parameter values.
KEYWORDS: Systems with delay; Stochatic processes; Wiener processes.

1. INTRODUCTION

Congestion control mechanisms and active queue management schemes (AQM) for the Internet have been extensively studied since the work of Kelly et all [C+04]. In [MO03], the Hopf bifurcation has been studied for the model of an Internet network with *r* (*r* >1) link and single source, which can be formulated as:

$$\dot{x}_i(t) = k\left(w - af\left(x_i(t-\tau)\right) - b\sum_{\substack{j=1 \\ j\neq 1}}^{r} x_j(t-\tau) f\left(x_j(t-\tau)\right)\right), i = 1, \dots, r \qquad (1)$$

where $x_i(t)$ is the sending rate of the source *i* at time *t*, *k* is a positive gain parameter, t is the sum of forward and return delays, *w* is a target (set –point), and the congestion indication function *f* (*x*) is increasing, nonnegative, which characterizes the congestion.
The discretizing model of (1) has been analyzed in [MO03], and the value of *k* has been determined, for which the Neimark-Sacker bifurcation takes place.
The dynamic model with a single link and *r* sources can be described by:

$$\dot{x}_i(t) = k_i x_i(t-\tau)\left(\frac{\alpha_i}{x_i(t)} - \beta_i x_i(t) p(t)\right), \dot{p}(t) = kp(t)\left(\sum_{i=1}^{r} x_i(t-\tau_i) - c\right), i = 1, \dots, r \qquad (2)$$

where, $x_i(t)$ is the rate at which source *i* transmits data at the time *t* , α_i and β_i are positive real numbers, *p*(*t*) is the loss probability function, τ_i is round-tripe delay for source *i* , *c* is the capacity, k_i, *k* is gain parameters.
The discretized model of (3) has been analyzed in [Moh98]. The models (1) and (2) lead to dynamic models described by stochastic delay differential equations, by randomizing one of the parameters. Thus, in section 2 we will consider stochastic delay differential equations, obtained through the randomization of parameter *a* in the case of the model (1), and, respectively, of parameters $\alpha_i, i = 1,\dots,r$ in the case of the model (2).
For certain functions which describe the stochastic delay differential equations (SDDE), the existence and uniqueness of the solution is justified.
In section 3, we describe the algorithm which approximates the solution of the equations for *r* =1, and, respectively, for the equations for *r*=3, in section 4, we describe numerical simulation, in section 5, we describe a part of the program in Maple 12 to get the graphical results and in section 6, the conclusions are presented.
For certain parameter values, we carry out numerical simulation with the Maple 12 software, using the Boxe-Muller method for the simulation of Wiener processes.

2. SDDE MODELS FOR INTERNET NETWORKS

Let (Ω, A, P) be a complete probability space with a filtration (A_t) satisfying the usual conditions; that is the filtration $(A_t)_{i\geq 0}$ is right - continuous and each (A_t), where $t \geq 0$ contains all P-null sets in *A*. For general theory we refer to [Tod92].

With $E(x)=\int_{\Omega} xdx$ we say for $1\le p\le\infty$ that $x\in L^p=L^p(\Omega,A,P)$ if $E(|x|^p)\langle\infty$ and we define $\|x\|_p=(E|x|^p))^{\frac{1}{p}}$. Here, E denotes the expectation.

Let $w(t)=(w_1(t),\ldots,w_r(t))^T$, $w(t)$ be a r - dimensional Wiener process given on the filtered probability space (Ω,A,P). The stochastic delay differential equation (SDDE), $(0=t_0\langle T\langle\infty)$ is given by:

$$\begin{aligned} &dx_i(t)=f_i(x(t),x(t-\tau))dt+g_i(x(t),x(t-\tau))dw_i(t),t\in[0,T]\\ &x_i(t)=\Psi_i(t),t\in[0,T],i=1,\ldots,r \end{aligned} \tag{3}$$

With one fixed lag, where $\Psi_i(t)$ is A_{t_0} - measurable, $C([-\tau,0],R)$.- value random variable so that $E(\|\Psi\|^2)\langle\infty$.

The system (3) can then formulated equivalently as:

$$x_i(t)=x_i(0)+\int_0^t f_i(x(s),x(s-\tau))ds+\int_0^t g_i(x(s),x(s-\tau))dw_i(s),i=1,..,r \tag{4}$$

for $t\in[0,T]$ and with $x_i(t)=\Psi_i(t)$, for $t\in[-\tau,0]$.

The second integral in (4) is a stochastic integral which can be interpreted according to ito's integral.

We have $f_i:R^r\times R^r\to R, g_i:R^r\times R^r\to R, i=1,\ldots,r$, and $\Psi_i:[-\tau,0]\to R, i=1,\ldots,r$, and and we will at various points, assume subset of the following set conditions:

1. The functions f_i and g_i are continuous;
2. The functions f_i and g_i satidfy a uniform Lipschitz conditions;
3. The functions Ψ_i is Holder - continuous with exponent Ψ_i;
4. The functions f_i and g_i satisfy a linear growth contition;
5. The partial derivatives of $f_i(\Phi,\Psi)$ exist and are uniformly bounded.

Proposition 1 [Tod92]. Assume that the functions f_i and g_i satisfy the assumption 1-3 above. Then there exists a unique strong solution equation (3).

The model for an Internet network with r ($r>1$) link and single source described by SDDE is given by:

$$\begin{aligned} &x_i(t)=x_i(0)+\int_0^t k(w-af(x_i(s-\tau))b\sum_{\substack{j=1\\ j\ne i}}^{n} x_i(s-\tau))f(x_i(s-\tau)))ds\\ &-\int_0^t kf(x_i(s-\tau))dw_i(s),t\in[0,T],i=1,..,r\\ &x_i(t)=\Psi_i(0),t\in[0,T],i=1,\ldots,r \end{aligned} \tag{5}$$

The model for the Internet network with a single link and r sources can be described by:

$$\begin{aligned} &x_i(t)=x_i(0)+\int_0^t k_i x_i(s-\tau)(\alpha_i g(x_i(s-\tau))-p_i x_i(s)f(s))ds,t\in[0,T],i=1,\ldots,r\\ &P(t)=P(0)+\int_0^t P(s)(\sum_{i=1}^{r} x_i(s-\tau)-c)ds\\ &x_i(t)=\Psi_i(0),t\in[-\tau,0]. \end{aligned} \tag{6}$$

If functions f, g and $\Psi_i, i = 1,\ldots,r$, satisfy conditions 1-5, then SDDE given (5) respectively (6) has a unique solution.
In what follows we will describe the algorithm that approximates the SDDE solution (5) with r =1 and $f(x) = \dfrac{a_1 x^2}{a_2 x + a_3}$, and solution SDDE of (6), with $g(x) = \dfrac{1}{x}$, with r=3.

3. NUMERICAL ANALYSIS FOR THE SDDE (5) AND (6)

The problem of solving an SDDE is reduced to one of solving a sequence of systems of SDDE of increasing dimension on successive intervals $[m\tau,(m+1)\tau]$. Using the method from [BB00, Tod92] for (5) with r =1 we obtain:

$$x_i(n+1) = x_i(n) + h(x(0) + k(w - af(x(n-p)) - kf((x(n-p)+A)\Gamma$$
$$-\frac{1}{2}kf(x(n-p)+A)f(x(n-p)+A)\Gamma^2, n \in N, p \in N \qquad (7)$$

where $A = \sqrt{\dfrac{1}{\alpha}}$, $\Gamma = [-4\alpha h \ln(\delta)]^{1/2}\cos(2\pi\delta)$, δ- random numbers uniformly distributed in interval (0,1) and $0 < h < 1$.
Using the method from [1,10] for (6) with $r = 3$, we obtain:

$$x_i(n+1) = x_i(n) + h(x_i(0) + k_i x_i(n-p)(\alpha_i g(x_i(n-p) - \beta_i x_i(n)P(n)) +$$
$$+k_i((x_i(n-p)+A_i)g(x_i(n-p)+A_i)\Gamma_i + {}^1\!/_2(x_i(n-p)+A_i)+$$
$$+(g(x_i(n-p)+A_i)+(x_i(n-p)+A_i)g(x_i(n-p)+A_i))\Gamma_i^2, i = 1,2,3 \qquad (8)$$
$$P(n+1) = P(n) + h(P(0) + P(n))(\sum_{i=1}^{3} x_i(s-\tau) - c), n \in N, p \in N$$

where $\Gamma_i = [-4\alpha_i h \ln(\delta_i)]^{\frac{1}{2}}\cos(2\pi\delta_i), \alpha_i \in (0,1), A_i = \sqrt{\dfrac{1}{\alpha_i}}$, and δ_i are are numbers uniformly distributed in the interval (0,1) . Using results from [1] we demonstrate that the algorithms (6) and (8) for f and g with properties 1-5 from section 2 are convergent.

4. NUMERICAL SIMULATION

For $\alpha_1 = 0.1, \alpha_2 = 0.5, \beta_1 = 0.1, \beta_2 = 0.1, c = 1, k_1 = 0.3, k_2 = 0.1$ and $g(x) = \dfrac{1}{x}$, in the following figures, Fig.1, Fig.2, Fig.3, Fig.4 we visualize orbits $(n, x_1(n,\omega))$, $(n, x_2(n,\omega))$, $(n, x_3(n,\omega))$ and $(n, P(\omega))$.

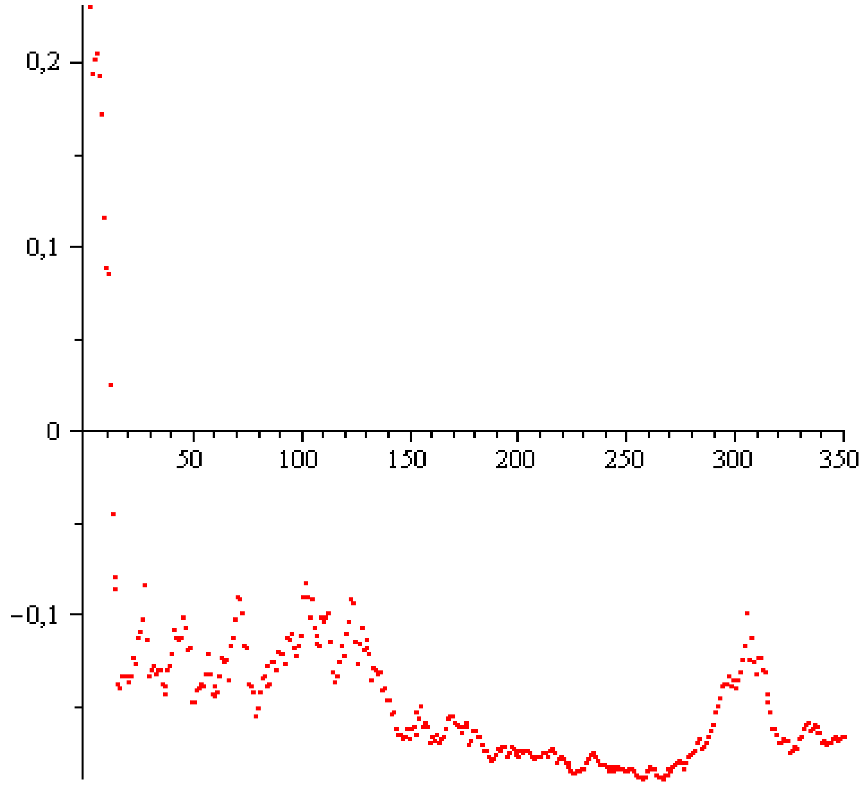

Fig.1 The orbit $(n, x_1(n,\omega))$

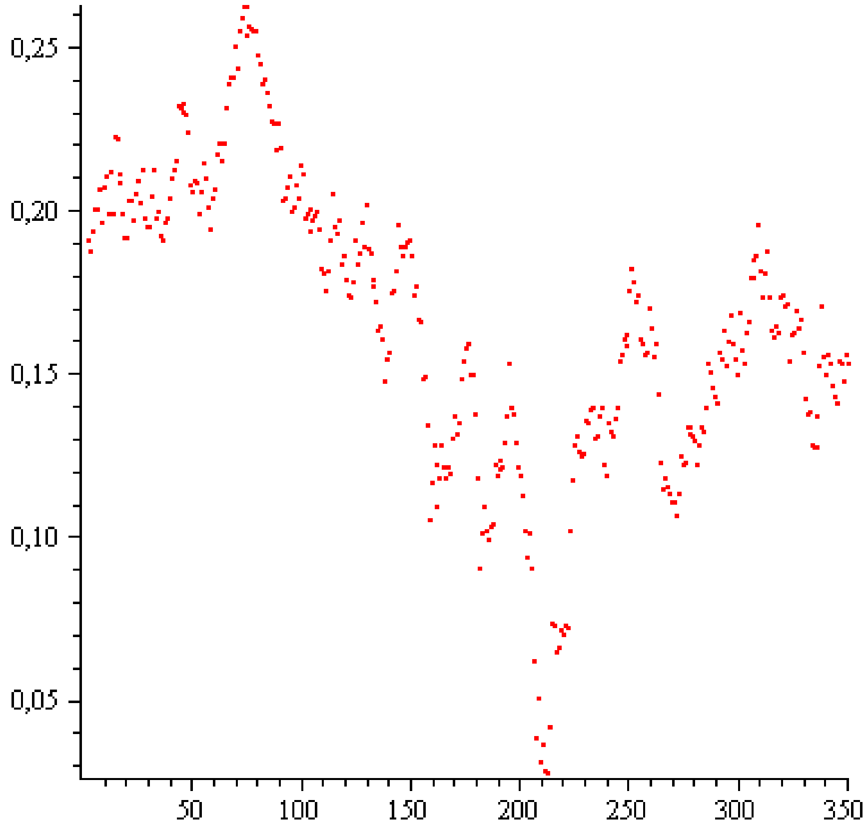

Fig.2 The orbit $(n, x_2(n,\omega))$

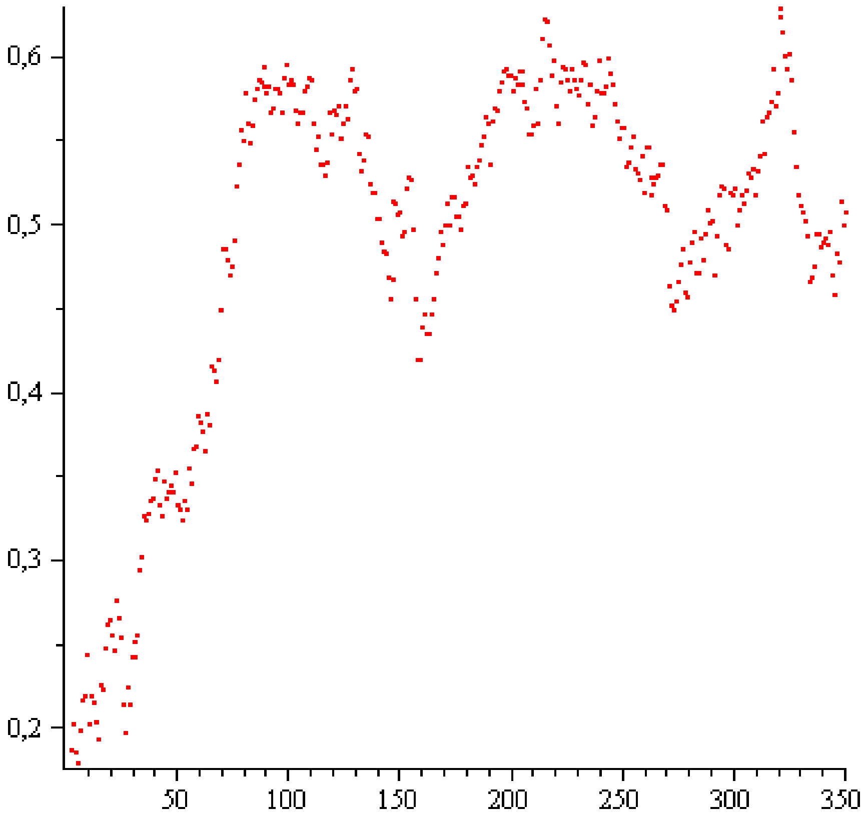

Fig.3 The orbit $(n, x_3(n, \omega))$

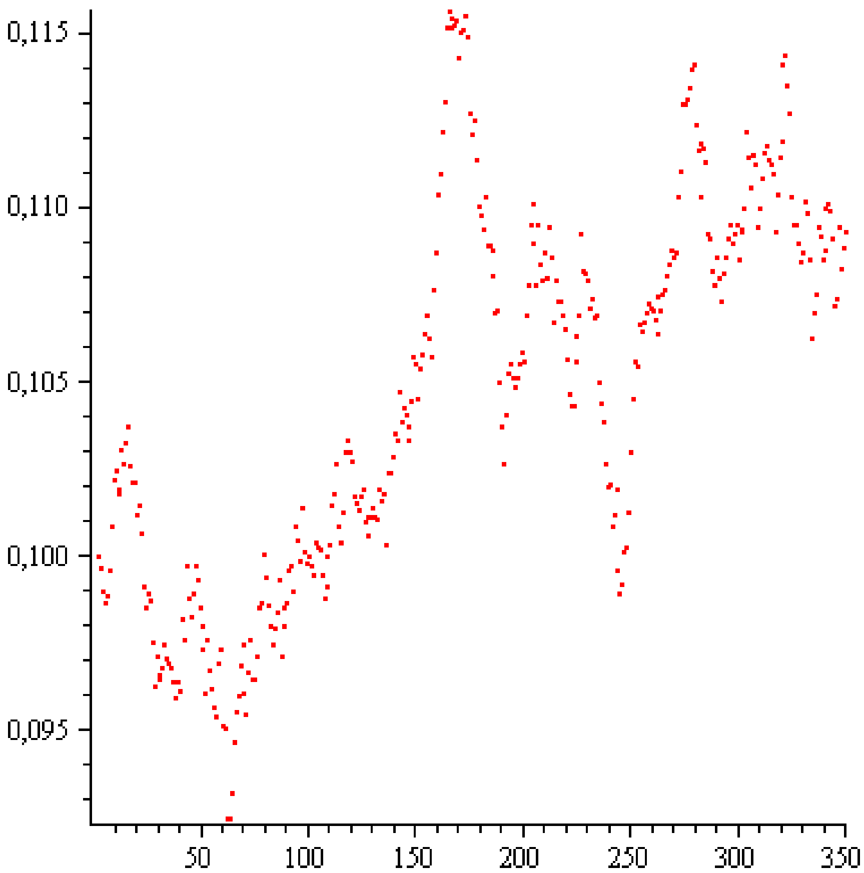

Fig.4 The orbit $(n, P(\omega))$

5. A PART OF THE PROGRAM TO GET THE GRAPHICAL RESULTS

The solutions have been visualized with a program in Maple 12, using the Boxe-Muller method for the simulation of Wiener processes.

```
> #Dynamic Models for Internet Networks Described by Stochastic
      Delay Differential Equations
> # SDDE Model for 3 resources and 1 link
> # Models' parameters: a1:=0.1; a2:=0.5; a3:=0.3; b1:=0.1; b2:=0.1; c:=1; k1:=0.3; k2:=0.1; g(x):=1/x; λ1:=0.2; λ2:=0.3; c:=1; k:=0.2;
>  a1:=0.1; a2:=0.5; a3:=0.1; b1:=0.1; b2:=0.02; b3:=0.3; k1:=0.3; k2:=0.1; k3:=0.2; k:=0.2, g(x):=1/x; c:=0.3;
> with(stats,random):
> T:=1.;n:=350;h:=T/n;
> x1:=array(-4..n):x2:=array(-4..n):x3:=array(-4..n):x1[-4]:=0.2;x1[-3]:=0.4;x1[-2]:=0.1:x1[-1]:=0.2:x1[0]:=0.1:x1[1]:=0.2:x2[-4]:=0.2;x2[-3]:=0.4;x2[-2]:=0.1:x2[-1]:=0.02:x2[0]:=0.1:x2[1]:=0.2:x3[0]:=0.02;x3[1]:=0.2;x4[0]:=0.2;x4[1]:=0.1;x3[-4]:=0.2;x3[-3]:=0.4;x3[-2]:=0.1:x3[-1]:=0.02;A1:=0.2;A2:=0.1;A3:=0.2;A4:=0.2;
> with(plots):
>
> for j from 2 by 1 to n do
>    x1[j]:=x1[j-1]+h*k1*a1*x1[j-2]/x1[j-1]-h*k1*b1*x1[j-1]*x1[j-2]*x4[j-1]+k1*(x1[j-2]+A1)*random[normald[0,sqrt(h)]](1)/x1[j-1];    x2[j]:=x2[j-1]+h*k2*a2*x2[j-3]/x2[j-1]-h*k2*b2*x2[j-1]*x2[j-3]*x4[j-1]+k2*(x2[j-3]+A2)*random[normald[0,sqrt(h)]](1)/x2[j-1]; x3[j]:=x3[j-1]+h*k3*a3*x3[j-3]/x3[j-1]-h*k3*b3*x3[j-1]*x3[j-3]*x4[j-1]+k3*(x3[j-3]+A3)*random[normald[0,sqrt(h)]](1)/x3[j-1];    x4[j]:=x4[j-1]+h*k*x4[j-1]*(x1[j-3]+x2[j-3]+x3[j-3]-c)-c*k*(x4[j-1]+A4)*random[normald[0,sqrt(h)]](1);    od:
> # The following images depict the orbits (n,x1(n,ω)), (n,x2(n,ω)), (n,x3(n, ω)) and (N,P(n, ω))
>
> plot({seq([j, x1[j]], j = 2..n)}, style = point, symbol = point, scaling
     = UNCONSTRAINED, caption
     = typeset("Fig.1 The orbit (n, x1(n,omega))"));

> plot({seq([j, x2[j]], j = 2..n)}, style = point, symbol = point, scaling
     = UNCONSTRAINED, caption
     = typeset("Fig.2 The orbit (n, x2(n,omega))"));

> plot({seq([j, x3[j]], j = 2..n)}, style = point, symbol = point, scaling
     = UNCONSTRAINED, caption
     = typeset("Fig.3 The orbit (n, x3(n,omega))"));

> plot({seq([j, x3[j]], j = 2..n)}, style = point, symbol = point, scaling
     = UNCONSTRAINED, caption
     = typeset("Fig.3 The orbit (n, P(n,omega))"));

>
```

6. CONCLUSIONS

This paper has introduced SDDE for an Internet network with a single source and *r* links, for an Internet network with *r* sources and one link. The paper has shown that these equations belong to the category of equations that accept a unique solution.

We have described a numerical algorithm in order to determine the approximate solution. The solutions have been visualized with the help of a program in Maple 12, using the Boxe-Muller method for the simulation of Wiener processes. A similar study will be conducted for cases in which other confidences will be randomized.

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