

MODELLING QUEUING SYSTEM WITH INVERSE GAMMA DISTRIBUTION: A SPREADSHEET SIMULATION APPROACH

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ABSTRACT: There is a need to provide user friendly approach to modeling and simulation for learners and business modeler. This study offers process-driven queuing simulation via spreadsheet which provides a user friendly, yet a readily available excel platform. Spreadsheet queuing simulation suggest a better way of understanding queue behavior than dedicated simulation software as it offers many benefits to students, practitioners and managers; helping them have firsthand experience with modeling. In this study, single server queue was simulated using Inverse Gama distribution. Generating single server queue (G/G/1) has been considered difficult, but this study presents easy to comprehend and applicable formulation. The spreadsheet simulation technique was applied to students queue at eatery in University of Lagos, Nigeria and results obtained were displayed and interpreted accordingly. This study recommends modelling and simulation of queuing systems with spreadsheet for students and business managers.

KEYWORDS: Queuing theory, Simulation, Spreadsheets, Inverse Gamma distribution, G/G/1.

1. INTRODUCTION

A queue is formed whenever current demand exceeds the existing capacity to serve, and the counter is so busy that arriving customers cannot receive immediate service in the facility, this require that the serving process be done as a queuing model. Spreadsheets are very user-friendly and yet powerful tool for modeling real business problems [Win04]. [Gro99] identified that building process-driven queuing simulation models have been found to be good end-user modeling exercises which help students understand about queues. The process-driven approach can present queue activities clearly and intuitively (for example, graphical representation in [IG02, Leo07]). Some authors have demonstrated and written about the use of the process-driven approach based on spreadsheets to explain queuing theory among which is ([Gro99, Eva00, IG02 and Hor03]. [Ban98], employed a process-driven approach to generate a table, called “ad hoc simulation table”, the table contains relevant information about each customer in a single- server

queuing system. The limitation of “ad hoc simulation table” is that it cannot be used to analyze some other variables like total number of customers in the system at a given time. [Pri98] and [CAJ98] provided an example of a more expounded table assigning the arrival and departure times in their proper chronological order which is called an “event calendar”. [Gro99] implemented queuing systems using only traditional spreadsheet functions with process-driven techniques by simulating a multi-server queuing system. The authors identified that the graphical display obtained in their implementation provides “a richer understanding of queues”. Also, [Win04] provided the event-driven approach to simulate single queuing systems.

[Liv93] studied single server queues with correlated arrival and service distributions which were generated by Transform Example Method (TES) [Mel91]. The simulation results of the system suggested that the mean waiting process to any node is usually approximated as a renewal process. However, in many circumstances, the accuracy of performance measures deteriorates whenever the correlations are significant ([Gir96, GH96]).

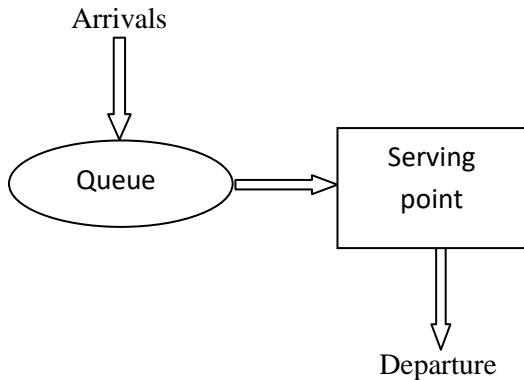
This study modeled and demonstrated the use of spreadsheets in carrying out basic queuing analysis using a fast food joint at Faculty of Education, University of Lagos, Nigeria. The eatery privately owned, located midway along the walk-way leading to the female student’s hostel from the Faculty of Education. It is a medium sized eatery which offers sales of different kind of drinks to compliment the toasted bread. It could be described as a monopolistic type of competition (one player). It runs from morning till night, and customer’s rush hour is always between 5pm to 10 pm after the day’s lecture.

The main purpose of this paper is to apply queuing theory to estimate the parameters. Therefore, a mathematical model using a spread-sheet formulation is developed for the analysis via spreadsheets. The remaining part of this paper is as follows, in section 2, Simple model for queue server is presented, section 3 is the practical application,

and in section, the results are presented. Finally, in section 5, the discussion and recommendation.

2. SIMPLE MODEL FOR SINGLE-QUEUE SERVER

This is a simple model for single-queue server:



2.1 The G/G/1 Queue

There are useful bounds being developed for waiting time W_q in a queue which can then be used to estimate other quantities, following model by Little that $W = W_q + \bar{S}_t$, where S_t is random service time.

$$E(S_t) = \bar{S}_t$$

$$\sigma_{S_t}^2 = [E(S_t^2)] - [E(S_t)]^2$$

Let λ = Average arrival rate of jobs

T_i = Inter-arrival time

$$E(T_i) = 1/\lambda$$

$$\sigma_{T_i}^2 = [E(T_i^2)] - [E(T_i)]^2$$

$\rho = \lambda \bar{X} =$ Traffic experienced, to have a stable queue, $\rho < 1$.

A G/G/1 queue, for a known mean and variance of the inter-arrival times and service times, the following bounds holds for the waiting time (W_q) in a queue.

$$\frac{\lambda \sigma_{S_t}^2 - \bar{S}_t(2 - \rho)}{2(1 - \rho)} \leq W_q \leq \frac{\lambda(\sigma_{S_t}^2 - \sigma_{T_i}^2)}{2(1 - \rho)} \quad (1)$$

Lower bound Upper bound

The result from lower bound is (1) above usually give insignificant conclusion, but the upper bound is to a large extent is useful.

G/G/1 offers a special case where the inter-arrival time T_i satisfies the following property for all values of t_i .

$$E(T_i - t_i | T_i > t_i) \leq \frac{1}{\lambda} \text{ for all } t_i \geq 0 \quad (2)$$

If the arrival progression follows (2), then the following bounds have been identified to hold [Bos02]:

$$W_{qU} - \frac{1 + \rho}{2\lambda} \leq W_q \leq W_{qU} \quad (3)$$

$$W_{qU} \text{ imply } \frac{\lambda(\sigma_{S_t}^2 - \sigma_{T_i}^2)}{2(1 - \rho)} \text{ given in (1)}$$

For a G/G/1 system, (3) can be used to find the bounds of mean number of customer waiting in queue (N_q). It expressed as:

$$\lambda W_{qU} - \frac{1 + \rho}{2\lambda} \leq N_q \leq \lambda W_{qU} \quad (4)$$

The difference between Lower and Upper bound is $0.5(1 + \rho)$. The upper bound in (1) offers heavy traffic approximation for the G/G/1 queue. As $\rho \rightarrow 1$, the traffic intensity increases, the distribution of the waiting time will be approximately an exponential distributed random variable [Bos02]. The result in (1) also provides a distribution for the waiting time in queue for a universal condition. In the Inverse gamma distribution is used since gamma distribution reduces to exponential distribution when $\alpha = 1$ in (5) below.

2.2 Inverse Gamma Distribution

Gamma distribution has probability density of:

$$f(x) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} \exp(-x/\beta) \text{ for } x > 0 \quad (5)$$

$\Gamma(1) = 1 \rightarrow$ exponential distribution

The Inverse Gamma (IG) distribution has the density:

$$f(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{-\alpha-1} \exp(-\beta/x) \text{ for } x > 0 \quad (6)$$

Relating Inverse gamma (IG) to gamma distribution, if X has a gamma (α, β) distribution, then

$Y = 1/X$ has an $IG(\alpha, 1/\beta)$ distribution. Using transformation theorem we have:

$$f_Y(y) = f_X(1/y) \left| \frac{d}{dy} y^{-1} \right| \quad (7)$$

$$= \frac{1}{\Gamma(\alpha)\beta^\alpha} y^{-\alpha+1} \exp(-1/\beta y) y^{-2} \quad (8)$$

$$= \frac{(1/\beta)^\alpha}{\Gamma(\alpha)} y^{-\alpha-1} \exp(-(1/\beta)/y) \quad (9)$$

Estimation of parameters follow the excel implementations.

3. PRACTICAL APPLICATION

To demonstrate the use of spreadsheets in implementing basic queuing process for G/G/1, data for first twenty customers was collected at the point of sales of the snack at University of Lagos, Nigeria. The eatery is located around the hostel and students are usually at the points mostly on returning from the day's lecture and would want to hang out with friends. The processing and service time varies because there are various options of the snack. Some customers want their snack(s) with egg, some with both fish and egg and so on. Table 1 shows the actual data collection starting from 5:21 pm.

Table 1: The inter-arrival time Arrival time and process time of customers

Event	Inter-Arrival Time (min:sec)	Cumulative Time	Actual time	Process Time
1	0	0	05:21:00	12:02
2	6:25	6:25	05:27:25	10:54
3	12:23	18:48	05:46:13	9:23
4	10:23	29:11	06:15:26	5:00
5	6:21	36:32	06:51:58	11:03
6	5:23	41:55	07:32:53	5:28

Source: authors' Computation

Table 1 first 6 observations of the 20 observations, Mean and standard deviation of inter-arrival is 7.85 (8.25) and 4.12 respectively.

3.1 The Excel GAMMA.INV Function

The Excel Gamma.Inv function is used for the purpose of this simulating the result. It is the inverse of the Excel Gammadist function. The (Excel 2010) obeys Gamma.Inv function while lower version obeys GammaInv. Gamma.Inv function has an improved accuracy over previous. It can be computed as Gamma.Inv (probability, alpha, beta).

To generate random numbers we have Gamma.Inv(Rand(), Alpha, Beta). After generating the model for the first row, copy, drag and paste on as many rows as desired. One thousand (1000) is considered in this study.

In order to carry out spreadsheet simulation on this data, computation such as the mean and standard deviation of inter-arrival time and service time was first computed which informs the value of α and β respectively.

Step 1: Compute mean and Standard deviation.

From the Figure 1 inter-arrival mean and standard deviation occupies E6 and F6 respectively. While the mean and standard deviation of 'service time' occupies E7 and F7 respectively.

Step 2: Compute parameters α and β :

Table 2: alpha and Beta computation from excel sheet

α	β
=E6^2/F6^2	=F6^2/E6
=E7^2/F7^2	=F7^2/E7

Step 3: Carry out the simulation and computations.

A total of 1000 customer arrival was simulated, for the Inter-arrival time we have

= Gamma.Inv(Rand(),\$H\$6,\$I\$6) because α and β occupies H6 and I6 respectively. Starting from B22, Copying B22 and select 1000 cells down the line and paste B22 cell copied will reproduce the Gamma generated inter-arrival time the click "calculate" at the bottom of the excel sheet to generate the numbers, otherwise, the copied cell might be replicated in all . Similar thing was done for others.

For Process time we have = Gamma.Inv(Rand(),\$H\$7,\$I\$7)

Customer Arrival Time= $C_{m-1} + B_m = C_{21} + B_{22}$

Start time = $\max(F_{m-1}, C_m) = \max(F_{21}, C_{22})$

Finish time = $D_m + E_m = D_{22} + E_{22}$

Time in queue = $D_m - C_m = D_{22} - C_{22}$

Time in System = $F_m - C_m = F_{22} - C_{22}$

Service time=Finish time-Start time

= $F_m - D_m = F_{22} - D_{22}$

Step 4: Compute results based on previous formulations (figure 4).

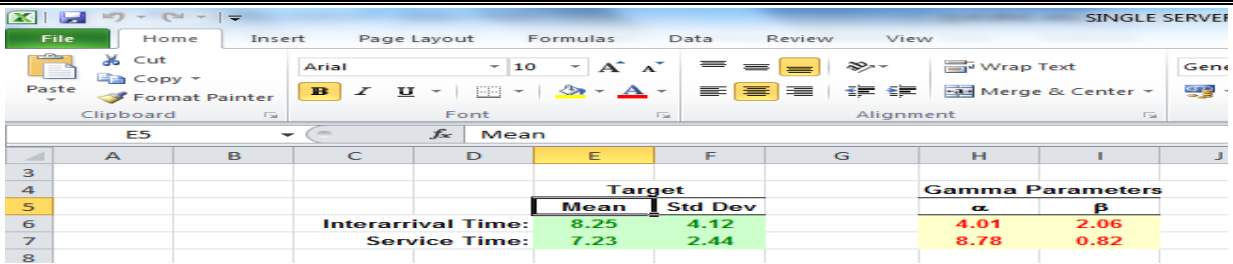


Figure 1: Formulation and computation of Alpha and beta

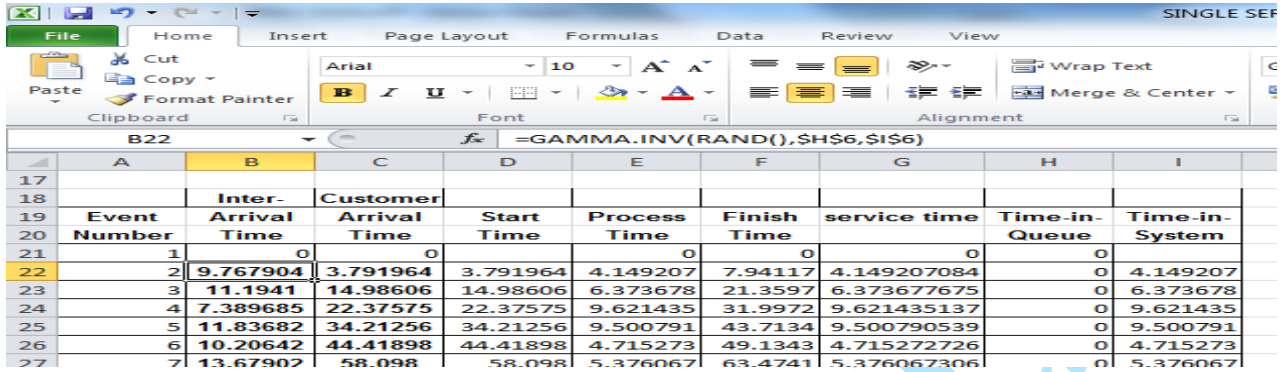


Figure 2: Formulation of simulated base Inter-arrival time and others

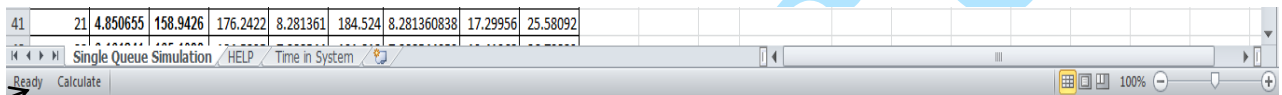


Figure 3: The computation point

	Simulation				Analytic
	Mean	Std Dev	Min	Max	Approx*
Utilization	90.58%				87.64%
Time in System (W)	19.24	13.35	1.15	64.06	12.06
Time in Queue (Wq)	12.03	13.15	0.00	57.83	11.24
Interarrival Time (a)	7.96	4.02	1.27	42.79	8.25
Process Time (p)	7.21	2.43	1.71	26.83	7.23

Figure 4: Final computations

3.2 Other Spreadsheet Computation

- i. Mean time in system (W) = AVERAGE (I22:I1021)
- ii. Mean time in queue (Wq)= AVERAGE (H22:H1021)
- iii. Mean interarrival time (a)= AVERAGE (B22:B1021)
- iv. Mean process time (p) = AVERAGE (E22:E1021)
- v. Standard deviation of time in system=STDEV (I22:I1021)
- vi. Standard deviation of time in queue= STDEV (H22:H1021)
- vii. Standard deviation of inter-arrival time= STDEV (B22:B1021)
- viii. Standard Deviation of Process Time= STDEV (E22:E1021)
- ix. Utilization= AVERAGE (I22:I1021)/ AVERAGE (B22:B1021)

- x. Analytical Approx* of time in queue

$$= \frac{0.5 * (F6^2 + F7^2)}{(E6 - E7)}$$

- xi. Analytical Approx* Utilization= E7/E6

- xii. Maximum time in System: TIS

$$\text{Min} = \text{MAX}(I22:I1021)$$

- xiii. Minimum time in System: TIS

$$\text{Max} = \text{MIN}(I22:I1021)$$

The methodology adopted in this research is the use of models that is underlying assumptions of queuing theory. The adoption of spreadsheet is to make simulation easily accessible to researchers in the area of simulation.

4. RESULTS

Records taken were tabulated and subjected to simulation using the excel spreadsheet. Details of summary of simulation can be found in Figure 4.

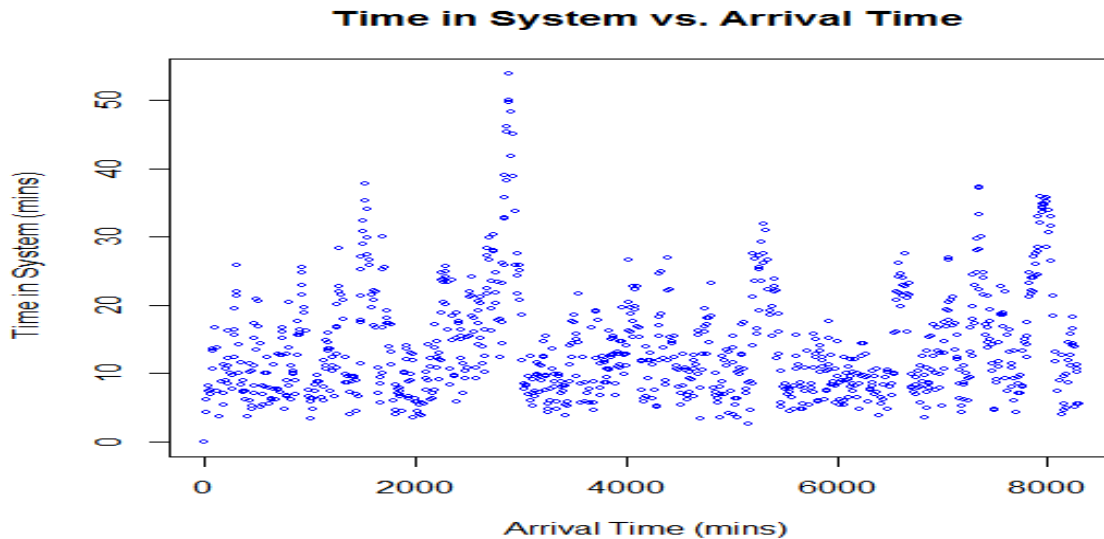


Figure 5: Showing time in system against Arrival time

Queuing Simulation

Mean time in queue (W_q) = 12.03 minutes

Mean time in system (W) = 19.24 minutes

Mean Inter-arrival time (a) = 8.30 minutes

Mean process time (ρ) = 7.24 minutes

Server Utilization = 87.17%

5. DISCUSSION AND RECOMMENDATION

This study shows practical application of modeling queuing using spreadsheet in the case of a single server busy queue, where 20 observations were made at initial stage. The mean and standard deviation of inter-arrival time and process time was used to compute the parameters Alpha and Beta of Gamma distribution. Thereafter, the excel spreadsheet was used for simulation of one thousand customers.

The simulation reveals that mean time in queue to be 12:03 minutes, mean time spent in the system is 19.24 minutes, mean inter-arrival time is 8:30 minutes and mean process time is 7.24 minutes and Server Utilization to be 87.17% which is close to 100%. The empirical analysis queuing system carried out may not be very efficient in terms of resources utilization. Queues form and customers wait even though server may be idle due to inefficient power supply and/or burnout on the part of the server. The fault is not in the model or underlying assumptions, it results from variability of the arrival and service process. If variability could be eliminated, systems could be designed economically so that there would be little or no waiting, and hence no need for queuing models.

As patronage for the snacks increases daily in the eatery, measure should be taken so as to avoid queues

which result in waste of time of customers (majorly students). Additional toasting machine and servers should be adopted, which may bring about reduction in queue length. The use of spreadsheet suggests a user friendly and accurate approach to modeling queue, it is therefore recommended for students, researcher and practitioner who require the use of available and efficient means of modeling queue.

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