DIFFERENTIAL EQUATION AS A SOLUTION FOR POPULATION PROBLEMS

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ABSTRACT: World of mathematical ideas, where a model is established, we then manipulate the model by the help of already existed techniques or computer numerical computations, finally we then investigate the model that will lead us to the solution to our mathematical problems related to the model, which is translated into a real life application. The application of first order differential equation in Growth and Decay problems will study the method of variable separable and the model of Malthus (Malthusian population model), where we use the methods to find the solution to the population problems which are of use in mathematics, physics and biology especially in dealing with problems involving growth and decay problems that requires the use of Malthus model.
KEYWORDS: Malthus model, Differential equation, Growth rate, Decay rate, Population rate, Population, Order, Attenuation.

INTRODUCTION

There are a lot of Ordinary Differential Equations (ODEs) that arises from different applications of mathematics. A first order differential equation is a differential equation that contains only derivatives of first order, and it has many applications especially in biology, engineering and many related mathematical related area of study. Some of the applications include population problem in the area of biology that requires the use of Malthus model.
According to some historians of mathematics the study of differential equations began in 1675, when Gottfried Wilhelm Von Leibniz wrote the equation:

$$\int x dx = \frac{1}{2}x^2 \qquad (1)$$

The research for general methods of integrating differential equation began when Isaac Newton classified first order differential equations into three classes.

$$\frac{dy}{dx} - f(x)$$
$$\frac{dy}{dx} - f(x,y) \qquad (2)$$
$$x\frac{du}{dx} + y\frac{du}{dx} - u$$

The first two classes contain only ordinary derivatives of one or more dependent variables with respect to single Independent variable and are known today as ordinary differential equations.
The Newton would express the right side of the equation in powers of the dependent an infinite series. The coefficients of the infinite number of particular solutions, it wasn't until the middle of 18th century that the full significance of this fact, i.e. that the general solution of a first order equation depends upon an arbitrary constant was realized.
Only in special cases can a particular differential equation be integrable in a finite form i.e. be finitely expressed in terms of known functions. In the general case one must upon solutions expressed in infinite series in which the coefficients are determined by recurrence formulae.
Karthikeyan and Srinivasan ([KS16]), studied the first order linear homogeneous differential equations and first order linear non-homogeneous differential equations and discovered that in the area of physics the first order linear homogeneous differential equations and first order linear non-homogeneous differential equations has a lot of applications in electrical circuits.
Longini *et al.* ([L+04]), studied the properties of using a targeted antiviral prophylaxis to carry an epidemic of flu before the active vaccine found. Using Malthusian mathematical population model and predicted that without any form of intervention an influenza illness can attack the host at a growth rate of about 33% of the entire population and an influenza death rate of about 0.58% per one thousand persons will result but with expected antiviral prophylaxis if 80% of the population is exposed and maintained on the antiviral prophylaxis for up to a certain number of weeks (8 weeks or above) the pandemics virus will be attenuated.
Colizza *et al.* ([C+07]), in their work: modeling the world wide spread of pandemic influenza baseline case containment intervention. Discovered that in the face of any threat of a new influenza pandemic due to H_sN_i avian flu virus that is commonly found in birds, and the fact to develop a new active vaccine if the pandemic happens to be in the population for

about 6 to 8 months, there is need to strategize an option of how to control the further spread of the virus before active vaccine is ready for distribution. By incorporating data on worldwide air travel and data from urban centers into mathematical model of the spread of influenza they were able to shoe that the rate of spread of the flu pandemic will depend among other factors on region then it arises from, the reproductive rate of the virus, if the reproductive of the virus rate (growth rate) is greater than 1.5 it will cause a severe pandemic. They also used their model to show that strict restriction on air travel will have little effect on the spread of the virus. They were also able to predict that if the R0 of the virus is up to 1.9 and a country stock provide enough antiviral drugs, enough to treat 5% of its population, the pandemic will be controlled.
Hassan and Zakari ([HZ18]) studied the first order ordinary differential equations and discovered that it has many application in temperature problems which leads to the use of Newton's law of cooling and arrives at the solution to the temperature problems that arise from first order differential equations.
The aim of this research is to study first order differential equations and use it to solve problems that arises in population growth and decay problems that requires the use of Malthusian population model.

1. CLASSIFICATION OF DIFFERENTIAL EQUATION

Differential equation is classified according to three properties which include;
i. Classification by type
ii. Classification by order and degree
iii. Classification as linear or non-linear differential equation.

1.1. Classification by Type

If an equation contains only ordinary derivatives of one or more dependent variables, with resects to a single independent variable is then called ordinary differential equation, ordinary differential equation in general any function of x_1 and the derivatives of y up to any order such that:

$$f\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3} \dots\right) = 0 \qquad (3)$$

Defines an ordinary differential equation for y (the dependent variable) in terms of x (the independent variable). For example

$$\frac{dy}{dx} - 3y = 0$$

is an example of ordinary differential equations.

1.2. Classification by Order and Degree

The order of differential equation is the highest differential coefficients contained in it.
Example

$$\frac{dy}{dx} - 3y = 0$$
$$(x + y)dx + 4ydy = 0$$

are examples of first order ordinary differential equations.
Equations of the form

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + y = sinx$$
$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 6y = 0$$

are examples of second order ordinary differential equations. Similarly equations

$$x\frac{du}{dx} + y\frac{du}{dy} = u$$
$$\frac{du}{dy} = -\frac{du}{dx}$$

are examples of first order differential equation.
The degree of a differential equation is the power to which the highest order differential coefficient is raised when the equation is rationalized (i.e. fractional power removed). In other words, the degree of any differential equation is the exponent of the highest powers for example the equation;

$$(\frac{dy}{dx})^4 + y^2 + (\frac{dy}{dx})^2 = 2$$
$$(\frac{d^2y}{dx^2})^2 + (\frac{d^2y}{dx^2})^5 + \frac{y}{x^2+1} = e^x$$

are of degree 4 and 2 respectively.

1.3. Classification As Linear Or Nonlinear Differential Equations

Definition: A linear (ODE) of order n, in the independent variable y is an equation that can be expressed in the form

$$a_0(x)y^n + a_1(x)y^{(n-1)} + \dots + a_{n-1}(x)y^1 + a_n(x)y = b(x) \qquad (4)$$

Where, $a_0(x) \neq 0$
And satisfies the following conditions
1) The dependent variable y and its derivative appear in the first degree only

2) No product of y and / or any of its derivatives appears in the equation
3) No transcendental function of y and / or its derivatives occur.

Examples:

(1) $\frac{d^3y}{dx^3} + 5x\frac{d^2y}{dx^2} + ysinx = 0$
(2) $cosxy'' + e^x y' - y = sinx$
(3) $x^2y'' + xy' + y = cosx$

In other hand a non linear ODE is a differential equation that violates one of the above conditions

(1) $\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^3 + 7y = 0$
(2) $\frac{d^2y}{dx^2} + y\frac{dy}{dx} = 0$

2. SOLUTION OF DIFFERENTIAL EQUATION

The first order differential equation has a variety of methods in finding its solution which includes the following,

a) Variable Separable
b) Equations Reducible To Variable Separable
c) Homogeneous Equations
d) First Order Exact Differential Equation
e) Linear Equations

(a) A first order differential equation can be solved by integration if it is possible to collect all y terms with dy, and all x terms with dx, i.e. if it is possible to write the equation in the form

$$f(y)dy + g(x)dx = 0. \qquad (5)$$

then the general solution is $\int f(y)dy + gxdx=c$, where c is an arbitrary constant.

(b) Equation reducible to variable separable, if we have a differential equation of the type

$$\frac{dy}{dx} = Ax + By + C \qquad (6)$$

Where $Ax + By + C$ is linear, i.e. to say the highest power of x and y is 1, and A, B and C are constants to the separable form. Homogeneous equation Equations that are not variable separable can be made so by a suitable change of variable, such equation is the one of the form

$$\frac{dy}{dx} = F\left(\frac{y}{x}\right) \qquad (7)$$

(c) Homogenous differential equation.
An equation of the form

$$\frac{dy}{dx} = \frac{f(x,y)}{\emptyset(x,y)} \qquad (8)$$

is homogenous if each $f(x,y)$and $\emptyset(x,y)$have the same degree example,

$$\frac{dy}{dx} = \frac{3xy + y^2}{3x^2 + xy}$$

(d) First order exact differential equation.
The equation of the form

$$Mdx + Ndy = 0 \qquad (9)$$

is an exact differential equation if $\frac{dM}{dy} = \frac{dN}{dx}$ where M is the coefficient of dx and N is the coefficient of dy.

(e) Linear equation, first order linear differential equation can be written in the form

$$\frac{dy}{dx} + Py = Q \qquad (10)$$

Where P and Q are function of x only, the way of solving this equation is the function $I = I(x)$ such that if we multiply both sides of the equation by I its left hand side **(LHS)** gives $\frac{d}{dx}(Iy) = IQ$

Then,

$$I\frac{dy}{dx} + IPy = \frac{d}{dx}(Py)$$

The right hand side **(RHS)**

$$\frac{d}{dx}(Iy) = I\frac{dy}{dx} + y\frac{dI}{dx}$$

Substitute the expression

$$I\frac{dy}{dx} + IPy = I\frac{dy}{dx} + y\frac{dI}{dx}$$

$$IPy = y\frac{dI}{dx}$$

$$IP = \frac{dI}{dx}$$

Using separation of variable

$$Pdx = \frac{dI}{I}$$

Integrating both sides of the equation we've

$$\int Pdx = \int \frac{dI}{I}$$

$$In = \int Pdx$$

$$I = e^{\int Pdx}$$

Hence I is known in any given question and also known as the integrating factor and is also known from equation

$$I\frac{dy}{dx} + IPy = IQ$$
$$I\frac{dy}{dx} + IPy = \frac{d}{dx}(Iy)$$

That is by equating the equation, thus

$$\frac{dy}{dx}(Iy) = IQ$$
$$\int \frac{dy}{dx}(Iy) = \int IQdx + c$$
$$Iy = \int IQdx + c \qquad (11)$$

3. GROWTH AND DECAY PROBLEMS IN DIFFERENTIAL EQUATION

In this age, we are aware of how infection by micro-organism such as Escherichia coli (E.coli) causes diseases. Many organisms (such as E.coli) produce a toxin that can cause sickness or even death. Some bacteria can reproduce in our bodies at a surprising fast rate, overwhelming our bodies' natural defenses with the sheer volume of toxin they are producing.
The rate at which the bacterial cultures grow is directly proportional to the current population (until such time as resources become scarce or overcrowding becomes a limiting factor). If we let P(t) to be the population at time t of any compartment say counting bacteria, e.t.c. Where P(t) is not continuous we need to determine the growth (input) rate and death (output) rate for the population.
Let us consider a population of bacteria that produce by simple cell division we assume that the growth rate is proportional to the population present this assumption is consistent with observation of bacteria growth provided there is food and space. We assume death rate is zero.
Therefore

$$\frac{dP}{dt} = kp \qquad (12)$$
$$P(0) = p_0$$
$$k > 0$$

(Growth rate p_o is at time t=0).

The solution of the equation $\frac{dP}{dt} = kp$ using variable separable we have

$$\frac{dP}{dt} = kp$$
$$dP = kpdt$$
$$\frac{dP}{p} = kdt$$

Integrate both sides of the equation

$$\int \frac{dP}{p} = \int kdt$$
$$inP = kt + c$$
$$P(t) = e^{(k+c)}$$
$$P(t) = e^{kt}e^{c}$$
$$P(t) = p_o e^{kt}$$

Where:

$$p_0 = e^c$$
$$P(t) = p_o e^{kt} \qquad (13)$$

Where P(t) is the population at any given time t, p_0 is the initial population, t is the time and k is a constant. Equation (13) is the Malthusian or exponential law of population growth.
Thomas Malthus (1766-1834) published a book on the principle of population as it affects the future improvements of the society in his book; Malthus put forth an exponential growth model for human population and concluded that eventually the population would exceed the capacity to grow an adequate food supply. We refer equation (13) as the general solution of the differential equation

$$\frac{dP}{dt} = kt$$

For $k > 0$, equation (13) is called an exponential growth law, and for $k < 0$, it is an exponential decay law.
A problem involving either decay or growth of a particular population requires the use of Malthusian population model (exponential growth or exponential decay rate).
EXAMPLE 1: A freshly incubated bacterial culture of streptococcus (a bacterium found in the throat and on the skin) contains 100 cells. When the culture is checked 60 minutes later, it is determined that there are 450 cells present. Assuming exponential growth:

(a) Determine the number of cells presents at any time t (measured in minutes)
(b) Find the doubling time;

SOLUTION: This type of problem requires the use of Malthusian population model (exponential growth rate) as stated from the problem.
(a) Using equation (13) for exponential growth rate we have,

$$P(t) = p_0 e^{kt}$$

Where P(t) is the population at any given time t, p_0 is the initial population, t is the time and k is a constant.
From the given data we have, P(0) = 100, (initial condition)
Setting P(t) = 100, and t = 0, we have

$100 = P_O \Rightarrow P_O = 100$

Now the above equation becomes

$P(t) = 100\ell^{kt}$ were k = constant

We use the second observation In order to determine the value of the growth constant k;
P(60) = 450 => t = 60 minutes
Therefore we have

$P(t) = P_O\ell^{kt}$
$450 = 100\ell^{k(60)}$

$$\frac{450}{100} = 1^{60k}$$

$$\frac{9}{2} = 1^{60k}$$

Take ln of both sides:

$$\ln\left(\frac{9}{2}\right) = 60k$$

$$k = \frac{1}{60}\ln\left(\frac{9}{2}\right)$$

k = 0.0251

We now have a formula representing the number of cells present at any time t as $P(t) = 100\ \ell^{(0.0251)\,t}$.
(b) The doubling time is

2P(t) = 2(100)
2P(t) = 20 so we have
$P(t) = P_o\ell^{kt}$

But $P_o = 100$,
And k = 0.0251
From the (a) above

$P(t) = 100\ell^{(0.0251)\,t}$

Put P(t) = 200
$200 = 100\ell^{(0.0251)\,t}$
$2 = \ell^{(0.0251)\,t}$
Take ln of both sides:

$$t = \frac{ln(2)}{0.0251}$$

t = 27.6
$\Rightarrow$ t = 28 minutes;

The population will be double when the time reaches 28 minutes.
From

$P(t) = 100\ell^{(0.0251)\,t}$
$P(28) = 100\ell^{(0.0251)\,(28)}$
P(28) = 201.

The bacterial culture will be double when the time is 28 minutes.
EXAMPLE 2: A bacteria culture is known to grow at a rate proportional to the amount present. After one hour, 1000 strands of the bacteria are observed in the culture and after four hours 3000 strands.

(a) Find an expression for the approximate number of strands of the bacteria presents in the culture at any time t. and
(b) Find the approximate number of strands of the bacteria originally in the culture.

SOLUTION:
(a) Let P(t) denote the number of bacteria strands in the culture at time t.
Then using exponential growth rate that is

$$\frac{dp}{dt} = kp$$

Which is both linear and separable, then we can use the method of separation of variables to find the solution

$P(t) = P_O\ell^{kt}$

At t = 1 hour,

P = 1000
$1000 = P_O\ell^{k(1)}$
$1000 = P_O\ell^{k}$ (14)

At t = 4 hours,

P = 3000
$3000 = P_O\ell^{k(4)}$
$3000 = P_O\ell^{4k}$ (15)

We solve equation (14) and (15) simultaneously to get

$K = {}^{1}/_{3}\ln(3)$
K = 0.3662

Then we use equation (14) to find P_o
$1000 = P_O\ell^{kt}$, but k = 0.3662
$1000 = P_O\ell^{0.3662}$

$(1000)\ \ell^{-0.3662} = P_o$
$P_o = 693$

Therefore the equation $P(t) = P_o\ell^{kt}$ now becomes:

$$P(t) = 693\ell^{(0.3662)t} \qquad (16)$$

The above equation is the expression for the approximate number of strands of the bacteria presents at any time t.
(b) To find the approximate number of strands of the bacteria originally in the culture.
We require P(t) at t = 0.
We substitute P(0) into the equation (16) we have:

$P(t) = P_O\ell^{kt}$
$P(t) = 693\ell^{(0.3662)(0)}$
$P(0) = 693\ P(t) = 693\ell^{0}$
$P(0) = 693$

Therefore the approximate number of strands of the bacteria originally in the culture is 693.
EXAMPLE 3: If it is assumed that the earth cannot support a population greater than twenty billion persons and that the rate of population growth is proportional to the difference between how closed the world population is to the limiting value, what is the mathematical expression describing the world population as a function of time?
SOLUTION: If P(t) is the world population according to the described model

$$\frac{dp(t)}{dt} = -k\left(p(t) - 20\right)$$

where k is negative to make $\frac{dp(t)}{dt}$ positive since P we must be less than 20 billion. Then solving the equation using variable separable we have

$$\frac{dp(t)}{p(t) - 20} = kdt$$

Integrate both sides of the equation

$$\int \frac{dp(t)}{p(t) - 20} = \int kdt$$

$$ln\left(p(t) - 20\right) = kt + c$$

$$\left|p(t) - 20\right| = p_0 e^{kt}$$

$P(t) – 20 \,|= P_o\ell^{kt}$,

Where $P_o = \ell^{C}$
Since P(t) is assumed less than 20 billion, thus

$-(P(t) – 20) = P_o\ell^{kt}$

By rearranging in order to make P(t) the subject we have:

$P(t) = 20 - P_o\ell^{kt}$

The above expression describes a world population as a function of time.

CONCLUSION

We have seen that the application of first order differential equations in decay and growth problems cannot be over emphasized especially in modeling the problems that deals with population of living organism and the most recommended model is by using Malthusian population model, and method of solution is preferably the separation of variable method.

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