CONSTRUCTION OF STEINER TRIPLE SYSTEM STS(2n+1) FROM A CLASS OF PAIRWISE BALANCED DESIGNS

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ABSTRACTS: Pairwise Balanced Design (PBD) is a pair (X, Ɓ) where X is a set of treatments and Ɓ is a collection of subsets of X called blocks, such that each pair of treatments is contained in precisely one block. PBD plays important role in design theories, it is used to construct other important designs such as Steiner Triple System (STS). A Steiner triple system is an ordered pair (*X*, Ɓ), where X is a finite set of points (Treatments) and *B* is a set of all 3-element subsets of *X* called triples, such that each pair of distinct elements of *X* occurs together in exactly one triple of *B*. The research work aims at applying a class of PBD(n, K, 1) when K ={3, 4} and λ = 1 to construct STS(2n +1). Theorem was proposed and proved and a certain inequality was derived as condition which must be satisfied for the construction to hold. Thus, for all n ≡ 1, 3(mod 3) of any PBD(n, {3, 4}) there exists an STS(2n +1) provided n ≥ l(s -1) + 1, where l is the size of the largest block of the PBD and s is the size of the smallest block of the PBD. Hence, STS(21) was constructed from a PBD(10, {3, 4}, 1).
KEYWORDS: *Block Designs, Pairwise Balanced Design (PBD), Steiner Triple System (STS) and Combinatorial System.*

1. INTRODUCTION

A PBD (n, K, λ) is a block design where n is the number of treatments, K= {k_1, k_2…, k_b} is the set of sizes of block and λ is number of time a pair of treatments appears together within blocks. The application of Pairwise Balanced Designs (PBDs) in the construction of related combinatorial systems is of paramount importance in design theory. PBD's were used in the construction of most resolvable and recursive designs. PBDs are block designs usually applied to achieved some balanced properties in experimental designs for example, PBD has it off diagonal elements matrix as constant which is the most important feature of a balanced designs. PBDs are also used to construct other designs such as Steiner Triple System, Pairwise Additive designs, Orthogonal Array, Partially Balanced Designs, Latin Square Designs etc. PBDs are also used to confirm the existence questions for other designs such as Balanced Incomplete Block Design (BIBD), Latin Square (LD), etc. Bose and Shrikhande ([BS60a]) defined Pairwise Balanced Design (PBD) as a pair (X, Ɓ) where X is a set of n-treatments and Ɓ is a collection of subsets of X called blocks, such that each pair of treatments is contained in precisely one block. Hanani ([Han63]) defined a PBD of index unity is a pair (X, Ɓ) where X is a set (of *points)* and Ɓ a class of subsets B of X (called *blocks)* such that any pair of distinct points of X is contained in exactly one of the blocks of Ɓ (and we may also require /B/≥ 2 for each B ϵƁ). Such systems are also known as linear spaces. PBD's where all blocks have the same size /B/ = k are known as Balanced Incomplete Block Designs (BIBD's) of index λ = 1, as 2 - (n, k,1) designs, and as Steiner Systems S(2, k,). The more general concept, where multiple block sizes are allowed, known as Pairwise Balanced Designs (PBD) was introduced by Bose and Shimamoto ([BS52]) and Hanani ([Han63]) had played important roles in their respective work on orthogonal Latin squares and BIBD's. The history of Steiner systems dates back to 1844, when Woolhouse studied triple systems, i.e., block designs with k = 3. The existence problem posed by him for triple systems was answered by Kirkman in 1847. This Steiner triple system has many connections with different areas and one of these areas is finite geometries. It is also known as the finite projective plane with smallest number (7 each) of points and lines, which is called the Fano plane ([LR08])

A Steiner triple system is an ordered pair (*X*, B), where X is a finite set of points (Treatments) and *B* is a set of all 3-element subsets of *X* called triples, such that each pair of distinct elements of *X* occurs together in exactly one triple of *B*.

For examples:

i). An STS(7) where n = 7 therefore,
X = {0, 1, 2, 3, 4, 5, 6} and B = {{0, 1, 3}, {1, 2, 4}, {2, 3, 5}, {3, 4, 6}, {4, 50}, {5, 6, 1}, {6, 0, 2}}
ii). An STS(3) where n = 3 (a trivial example) therefore,

X = {0,1,2} and B = {0, 1, 2}
iii). An STS(1) where n = 1 (a more trivial example) therefore,
X = {1} and B = Ø

A resolvable *STS*(n) is called a Kirkman triple system, and it is denoted by KTS(n). The solution to Kirkman's 15 schoolgirl problem is a resolution of a block design on 15 points with blocks of size 3 such that every pair of points is contained in a unique block. A resolvable Steiner Triple System is called a Kirkman Triple System and there are 7 non-isomorphic Kirkman Triple Systems on 15 points'(n). The first person who mentioned resolvable designs was Thomas Kirkman ([Kir47]). He asked the following question, known as Kirkman's schoolgirl problem: Fifteen young ladies in a school walk out three abreast for seven days in succession: It is required to arrange them daily, so that no two shall walk abreast. In other words, the problem is equivalent to finding a KTS(15). If the girls are numbered from 01 to 15, one of the seven solutions is given below: below, where the schoolgirls are labeled 1, 2, ... , 15:

Table 1: Kirkman's schoolgirl problem's results

DAY 1	DAY 2	DAY 3	DAY 4	DAY 5	DAY 6	DAY 7
1,6,11	1,8,10	1,3,9	1,2,5	2,3,6	1,7,14	1,12,13
2,7,12	2,9,11	2,13,14	3,10,12	5,7,13	3,5,11	2,4,10
3,8,13	3,4,7	4,5,8	4,11,13	8,9,12	4,6,12	5,6,9
4,9,14	5,12,14	6,7,10	6,8,14	10,11,14	9,10,13	7,8,11
5,10,15	6,13,15	11,12,15	7,9,15	1,4,15	2,8,15	3,14,15

It is easily seen that the above plan is a BIBDs with parameters n = 15, *b* = 35, *r* = 7, *k* = 3, and λ = 1 when triplets of girls are treated as blocks. Other solutions to KTS{15) were provided by several authors, including Cayley ([Cay50]), Peirce ([Pei60]) and Davis ([Dav97]). The solution of a Kirkman triple System KTS(n) for all n ≡ 3 (mod 6) was provided by Raychaudhuri and Wilson ([RW71]). Steiner ([Ste53]) proposed the problem of arranging *n* objects in triplets (called Steiner's triple systems) such that every pair of objects appears in exactly one triplet. It is easy to see that Steiner's triples are in fact BIB designs with block size three.
Later in 1853, Steiner discussed t -designs with k = t + 1 and λ = 1. When t = 2, these are triple systems with λ = 1. Thus, we call such designs Steiner triple systems. Put another way, they are 2-designs with parameters (n, 3, 1), and we denote such systems by *S*(2, 3, n). Unaware of Kirkman's work, Jakob Steiner ([Ste53]) reintroduced triple systems, and as this work was more widely known, the systems were named in his honor ([Ste53]).The existence question for which STS(n) does exist was first posed by W.S.B. Woolhouse (Prize question 1733, Lady's and Gentlemen's Diary 1844).The problem was solved in 1847 by Rev. T.P. Kirkman ([Kir47]) further established the existence of a Steiner triple system of order n which exists if and only if n≡ 1 , 3 (mod 6).

2. THE STEINER TRIPLE SYSTEM AND PBD

Wilson ([Wil74]) and Hanani ([Han79]) uses PBD's and GDD's to construct Steiner systems STS(2, ***k*, *n*).** The final aim is to facilitate the construction of Steiner systems STS (2n+1***)*** and it is with this in mind that we attempt a generalization of the fundamental construction.
A necessary and sufficient condition for the existence of an STS (l, K, n) is that n has an expression as a sum of members of K. A weaker condition, n≡ 0 (mod g.c.d.(K)), is sufficient for large values of n.
Necessary conditions for the existence of an S(t, K, n)with t> 1, are:
The existence of an S(t- 1, K- 1, n- 1) (where K- 1 denotes the set{k- 1: k∈K}), and

$\binom{n}{t} \equiv 0(\,mod\; g.c.d\binom{k}{t})$ where k∈ *K*

Wilson ([Wil75]) was able to show that these two conditions are sufficient for the existence of an S(2, K, n) provided n is sufficiently large. One would like to prove a similar theorem for S(3,K, n), i.e. when n is sufficiently large, the necessary congruence conditions are sufficient for the existence of an S(3, K, n). Hanani ([Han79]) has shown that the necessary conditions given above for the existence of S(3,4, n) and S(3, {4,6}, n) are indeed sufficient. Hanani ([Han79]) and Hanani ({Han71]) also proved the result for some larger sets of block sizes. The paper of Hanani ([Han79]) is also the state of the art as regards the existence of S(3, k, n) with k ≥ 5, except for a few constructions of individual designs which are listed in ([BJL86]).

2.1. Necessary Condition for the Existence of STS(n)

Using the same arguments as for Balanced Incomplete Block Designs (BIBD), one could conclude that for an STS(n), we have point replication and number of blocks given by

$r = \frac{n-1}{2}$ and
$b = \frac{n(n-1)}{6}$

Since r is an integer, we must have $\frac{2}{n-1}$, i.e n odd.
So n ≡1, 3, 5 (mod 6).
Since b is an integer, we must have $\frac{6}{n(n-1)}$.
For n ≡ 5 (mod 6), we cannot have $\frac{6}{n(n-1)}$.

Proposition (Necessary conditions for the existence of an STS(n)
If an STS(n) exists, then n ≡ 1, 3 (mod 6).

3. THE CONSTRUCTION

Recall that **a** Pairwise Balanced Design (PBD) is a pair (X, Ɓ) where X is a set of *n*- treatments and Ɓ is a collection of subsets of X called blocks, such that each pair of treatments is contained in precisely one block. Also, the balance properties of BIBDs can be extended to larger sets of elements in the form of PBDs by relaxing the requirement that all blocks are the same size, then we shall have a pairwise balanced design.

For example:

A (n, k, λ)-BIBD is a PBD(n, {k}, λ).
Then, If 2 ∈ K, a PBD(n, K, λ) exists for all n, K, and λ.
Let K ≠ {n}. If a PBD(n, K, λ) exists then n ≥ *l(s – 1)*+ 1 where *l* and *s* are the sizes of the largest and smallest blocks of the PBD, respectively.

Proof.

Let $B = \{n_1,...,n_l\}$ be a block of size *k*. Now $l<n$ so there exists n_0 in *X* such that $n_0 \notin B$. Since $n_0$must be paired with $n_1,...,n_l$ there exists a unique block B_i containing n_0 and n_i for each *i* from 1 to *l*.
Now in each B_i there are at least *s*−1 elements besides n_0, where $\cap B_i = \{n_0\}$.
Therefore,

$$n \geq \left|\bigcup_{i=1}^{l} Bi\right| = |\{x0\}| + \left|\bigcup_{i=1}^{l} Bi - \{x0\}\right| = 1 + \sum_{i=1}^{l} |Bi - \{x0\}| = l(s\text{-}l) + 1$$

For example:

There cannot exist a PBD(8,{3,4},1). If there are no blocks of size 4, we are looking for an STS(8); however, 8 is not congruent to1, 3 (mod 6). If there is at least one block of size 4, then it will give $n \geq 4(3-1)+1 = 9$.
Suppose there exists a PBD(*n*, K, λ_1) and for all k ∈ K there exists a PBD(k, L, λ_2). Then there exists a PBD(*n*, L, $\lambda_1\lambda_2$).

Proof:

Consider a PBD(n, K, λ_1) with blocks $B_1,...,B_n$ where B_i has k_i elements. Replace each b_i with the blocks of a PBD(k_i, L, λ_2), where instead of using the elements $\{1,...,k_i\}$ we use the elements of B_i according to some bijection $f_i : \{1,...k_i\} \longrightarrow B_i$. One claims that this is a PBD($n, L, \lambda_1\lambda_2$).
Trivially. there are *n*-treatments and the block sizes come from *L*. It suffices to show that any pair of treatments occurs exactly $\lambda_1\lambda_2$ times.
Consider two treatments *x* and *y*. They occur together in λ_2 blocks of the design each time they occurred in a block B_i of the original design. Since they occurred together in λ_1 blocks of the original design, the new design contains the pair (*x*, *y*) exactly $\lambda_1\lambda_2$ times. That is, the proposed design is a PBD($n, L, \lambda_1\lambda_2$) as desired.

For example:

Consider a PBD(10, {3, 4}) with the following blocks:

{1, 2, 3, 4}, {1, 5, 6, 7}, {1, 8, 9, 10},
{2, 5, 8}, {2, 6, 9}, {2, 7, 10}, {3, 5, 10},
{3, 6, 8}, {3, 7, 9}, {4, 5, 9}, {4, 6, 10}, {4, 7, 8}.

One could have PBD(3, {3}, 2) (a double "STS3", given by {1, 2, 3}, {1, 2, 3}) and a PBD(4, {3}, 2) i.e ((4, 3, 2)-BIBD which is given by {1, 2, 3}, {1, 2, 4}, {1, 3, 4}, {2, 3, 4}).
One could use these to obtain a PBD(10, {3}, 2) which is a (10, 3, 2)BIBD
Which implies that a PBD(10, {3}, 2) gives a (10, 3, 2)-BIBD.
Thus, the following blocks

{1, 2, 3, 4} → {1, 2, 3}, {1, 2, 4}, {1, 3, 4}, {2, 3, 4}
{1, 5, 6, 7} → {1, 5, 6}, {1, 5, 7}, {1, 6, 7}, {5, 6, 7}
{1, 8, 9, 10} → {1, 8, 9}, {1, 8, 10}, {1, 9, 10}, {8, 9, 10}
{2, 5, 8} → {2, 5, 8}, {2, 5, 8}
{2, 6, 9} → {2, 6, 9}, {2, 6, 9}
{2, 7, 10} → {2, 7, 10}, {2, 7, 10}
{3, 5, 10} → {3, 5, 10}, {3, 5, 10}
{3, 6, 8} → {3, 6, 8}, {3, 6, 8}
{3, 7, 9} → {3, 7, 9}, {3, 7, 9}
{4, 5, 9} → {4, 5, 9}, {4, 5, 9}
{4, 6, 10} → {4, 6, 10}, {4, 6, 10}
{4, 7, 8} → {4, 7, 8}, {4, 7, 8}.

If there exists a PBD(n, K, λ_1) and a (n', l, λ_2)-BIBD for all $n' \epsilon$ K, then there exists a (n, l, $\lambda_1\lambda_2$)-BIBD; and also, If there exists a PBD (n, K, 1) and as STS(n') for all $n' \epsilon$ K, then there exists an STS(n).

The derived theorem 3

For all n ≡ 1, 0 (mod 3) of any PBD(n, {3, 4}) there exists an STS(2n +1) provided n ≥ l(s -1) + 1, where l is the size of the largest block of the PBD and s is the size of the smallest block of the PBD

The proof:

Consider a PBD(n, {3, 4}, 1) a new design D can be constructed from the PBD as follows:
STEP 1. For each element e of the PBD include the block $\{\infty, e_0, e_1\}$
STEP 2. For each block $\{x, y, z\}$ (when the block size k = 3) of the PBD include the blocks $\{x_1, y_1, z_1\}$, $\{x_1, y_0, z_0\}$, $\{x_0, y_1, z_0\}$, $\{x_0, y_0, z_1\}$, which with $\{\infty, x_0, x_1\}$, $\{\infty, y_0, y_1\}$, and $\{\infty, z_0, z_1\}$ e.g this completes anSTS(7) on $\{\infty, x_0, x_1, y_0, y_1, z_0, z_1\}$.
STEP 3. For each block $\{a,b,c,d\}$(when the block size k = 4) of the PBD include the blocks $\{a_1,b_1,c_1\}$, $\{a_1,b_0,d_1\}$, $\{a_1,c_0,d_0\}$, $\{a_0,b_1,d_0\}$, $\{a_0,b_0,c_0\}$, $\{a_0,c_1,d_1\}$, $\{b_1,c_0,d_1\}$, $\{b_0,c_1,d_0\}$. which with $\{\infty,a_0,a_1\}$, $\{\infty,b_0,b_1\}$, $\{\infty,c_0,c_1\}$, and $\{\infty,d_0,d_1\}$ completes and STS(9) on $\{\infty,a_0,a_1,b_0,b_1,c_0,c_1,d_0,d_1\}$.
The claim that $D$is an STS (2n+1).
Trivially, D is defined on $2n$ +1 elements, each block is a triple, and the triples are incomplete. It remains to show that each pair of elements in D occurs in exactly one triple. We consider three cases.

1. Pairs of the form (∞, e_i). These are contained exactly once in the unique triple $\{\infty, e_0. e_1\}$.

2. Pairs of the form (e_0, e_1). These are contained exactly once in the unique triple $\{\infty, e_0. e_1\}$.

3. Pairs of the form (α_i , βj). Since α and β occur together in exactly one block of the PBD, α_i and β_j can be found together exactly once in the STS (7) or STS(9) which corresponds to that block.

That is, each pair of elements is found together in exactly one triple, so $D$is an STS ($2n$+ 1).

For example:

Using a PBD(10, {3, 4}, 1) = STS(21) which satisfied the following conditions of the derived theorem.
i). n ≡ 0, 1 (mod 3)
ii). 3 ≤ k ≤ 4
iii). λ = 1
iv). n ≥ l(s -1) + 1, where l is the size of the largest block of the PBD and s is the size of the smallest block of the PBD
therefore, an STS(21) can be constructed as follows:

STEP 1. For each element e of the PBD include the block $\{\infty, e_0, e_1\}$ Thus, The PBD (10, {3, 4}, 1) has n = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}, with k = {3}, then one could have the following pairs:
$\{\infty,1_0,1_1\},\{\infty,2_0,2_1\},\{\infty,3_0,3_1\},\{\infty,4_0,4_1\},\{\infty,5_0,5_1\},$
$\{\infty,6_0,6_1\},\{\infty,7_0,7_1\},\{\infty,8_0,8_1\},\{\infty,9_0,9_1\},\{\infty,10_0,10_1\}$

STEP 2. For each block $\{a, b, c, d\}$ when the block size k = 4 of the PBD include the blocks $\{a_1, b_1, c_1\}$, $\{a_1, b_0, d_1\}$, $\{a_1, c_0, d_0\}$, $\{a_0, b_1, d_0\}$, $\{a_0, b_0, c_0\}$, $\{a_0, c_1, d_1\}$, $\{b_1, c_0, d_1\}$, $\{b_0, c_1, d_0\}$. which with $\{\infty,a_0,a_1\}$, $\{\infty,b_0,b_1\}$, $\{\infty,c_0,c_1\}$, and $\{\infty,d_0,d_1\}$.
Listing all the blocks of the design we shall have:

{1, 2, 3, 4} $\{1_1, 2_1, 3_1\}$, $\{1_1, 2_0, 4_1\}$, $\{1_1, 3_0, 4_0\}$, $\{1_0, 2_1, 4_0\}$
$\{1_0, 2_0, 3_0\}$, $\{1_0, 3_1, 4_1\}$, $\{2_1, 3_0, 4_1\}$, $\{2_0, 3_1, 4_0\}$
4 counts

{1, 5, 6, 7} $\{1_1, 5_1, 6_1\}$, $\{1_1, 5_0, 7_1\}$, $\{1_1, 6_0, 7_0\}$, $\{1_0, 5_1, 7_0\}$
$\{1_0, 5_0, 6_0\}$, $\{1_0, 6_1, 7_1\}$, $\{5_1, 6_0, 7_1\}$, $\{5_0, 6_1, 7_0\}$
4 counts

{1, 6, 9, 10} $\{1_1, 8_1, 9_1\}$, $\{1_1, 8_0, 10_1\}$, $\{1_1, 9_0, 10_0\}$, $\{1_0, 8_1, 10_0\}$
$\{1_0, 8_0, 9_0\}$, $\{1_0, 9_1, 10_1\}$, $\{8_1, 9_0, 10_1\}$, $\{8_0, 9_1, 10_0\}$
4 counts

STEP 3. For each block $\{x, y, z\}$ (when the block size k = 3) of the PBD include the blocks $\{x_1,y_1,z_1\}$, $\{x_1,y_0,z_0\}$, $\{x_0,y_1,z_0\}$, $\{x_0,y_0,z_1\}$.Thus, listing all the blocks of the design we shall have:

{2, 5, 8} $\{2_1, 5_1, 8_1\}$, $\{2_1, 5_0, 8_0\}$, $\{2_0, 5_1, 8_0\}$, $\{2_0, 5_0, 8_1\}$
1 count

{2, 6, 9} $\{2_1, 6_1, 9_1\}$, $\{2_1, 6_0, 9_0\}$, $\{2_0, 6_1, 9_0\}$, $\{2_0, 6_0, 9_1\}$
1 count

{2, 7, 10} $\{2_1, 7_1, 10_1\}$, $\{2_1, 7_0, 10_0\}$, $\{2_0, 7_1, 10_0\}$, $\{2_0, 7_0, 10_1\}$,
1 count

{3, 5, 10} $\{3_1, 5_1, 10_1\}$, $\{3_1, 5_0, 10_0\}$, $\{3_0, 5_1, 10_0\}$, $\{3_0, 5_0, 10_1\}$,
1 count

{3, 6, 8} $\{3_1, 6_1, 8_1\}$, $\{3_1, 6_0, 8_0\}$, $\{3_0, 6_1, 8_0\}$, $\{3_0, 6_0, 8_1\}$

1 count

{3, 7, 9} $\{3_1, 7_1, 9_1\}$, $\{3_1, 7_0, 9_0\}$, $\{3_0, 7_1, 9_0\}$, $\{3_0, 7_0, 9_1\}$

1 count

{4, 5, 9} $\{4_1, 5_1, 9_1\}$, $\{4_1, 5_0, 9_0\}$, $\{4_0, 5_1, 9_0\}$, $\{4_0, 5_0, 9_1\}$

I count

{4, 6, 10} $\{4_1, 6_1, 10_1\}$, $\{4_1, 6_0, 10_0\}$, $\{4_0, 6_1, 10_0\}$, $\{4_0, 6_0, 10_1\}$

1 count

{4, 7, 8} $\{4_1, 7_1, 8_1\}$, $\{4_1, 7_0, 8_0\}$, $\{4_0, 7_1, 8_0\}$, $\{4_0, 7_0, 8_1\}$.

1 count

21 in all.

4. CONCLUSION

The research shows that PBD(n, K, 1) when K= {3, 4}, and λ = 1could be used to construct an STS(2n +1) provided $n \equiv 0, 1 \pmod 3$ and $n \geq l(s -1) + 1$. where l and s are the largest and smallest block sizes of the PBD.

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