NUMERICAL SOLUTIONS TO THE EXTINCTION PROBLEM OF POLIO TRANSMISSION AGENTS IN NIGERIA

Johnson Gbola Olawuwo, Olabisi O. Ugbebor

Department of Mathematics, University of Ibadan, Nigeria

Corresponding Author: Johnson Gbolagunte Olawuwo, gbola.2000@yahoo.com

ABSTRACT: In spite of local and international efforts to end the scourge of poliomyelitis in the world, the disease still exists around the world with new cases being detected in otherwise polio- free areas while the known endemic places of Asia and Sub-Saharan Africa are yet to be ridden of the epidemic. Knowing with accuracy whether the disease has finally ended in a population has always being a subject of interest to mathematicians, statisticians and epidemiologists. In this study, we have defined a conditional probability function $x_{ic}(t)$ as the probability of zero transmission agents given that there are initially (i) symptomatic (ii) infectives and (iii) asymptomatic carriers. Using the properties of Poisson process, two backward equations were derived which were jointly solved numerically. With these numerical solutions, we were able to formulate a model for the two identified transmission agents of poliomyelitis.

KEY WORDS: Modelling, Backward Equation, Picard's Method, Polio-free Period, Polio Virus.

INTRODUCTION

Although the severity of poliomyelitis in Nigeria has abated in recent times compared to the experience of the pre- 2015 era, yet there is still much to be done to completely wipe out the disease from Nigerian population.

Since the country's failed attempt at getting certified polio-free by the World Health Organisation in 2016, subsequent reports have indicated discovery of many new cases in some parts of the country showing that the hope of eradicating the disease in a near future is very bleak if drastic actions are not taken to reverse the trend.

Though, a lot of success has been recorded in the nation's anti-polio campaign much of these local and international efforts have met a lot of resistance arising from religious and cultural practices, accessibility of health workers to susceptible groups, insurgency problem and lack of commitment on the part of some local health workers. The problem of polio transmission is not limited to Nigeria but also applies to some other developing countries. According to the Global Polio Eradication Initiative ([***19]), new cases have been reported in Congo Democratic Republic and Niger in 2018. New cases were also recorded in Papua New Guinea in 2018 after 20 years without polio. Mozambique also had 2 new cases of vaccine derived polio virus in January 2019.

Most important to this study is the problem of monitoring and accurately assessing the degree of success or failure of the disease control at any point in time. Our interest therefore is to find an effective means of determining how far the two transmission agents have been eliminated from the population using numerical solutions of some associated stochastic differential equations which are related to those earlier used by Olawuwo and Ugbebor ([OU18]).

Among the existing works on poliomyelitis is the work of Dietz and Eichner ([DE96]) where simulation method was used to estimate the length of time without observing symptomatic cases to be 95% sure that poliomyelitis has been eradicated. Olawuwo and Ugbebor ([OU16]) used matrix exponential method to find that asymptomatic carriers of poliomyelitis have the capability to cause re-emergence of the disease with probability 0.0322. Also, after identifying two types of polio transmission agents as infectives and carriers, Olawuwo and Ugbebor ([OU18]) derived a pair of stochastic equations whose analytical solutions were used in modelling the extinction probability of the transmission agents and thereby setting the condition for its eradication. In their study of poliomyelitis, Denes et al ([DS17]) et al used persistent theory of Lyapunov function to propose a compartmental model. While studying the potential consequences of re-introduction of polio for different post eradication experience, Radbond et al ([RDM06]) found that re- emergence may arise from laboratory, inactivated polio vaccine manufacturing sites or bioterrorism

In this work, we have developed another pair of stochastic differential equations describing a no-incidence period given that the period started with as few as only one transmission agent which is either an infective or an asymptomatic carrier. These equations are jointly solved numerically and their solutions combined for a more comprehensive model for extinction probability. It is hoped that the model shall go a long way in monitoring polio eradication efforts with inputs from results of further clinical researches.

METHODOLOGY

Let D(t) be the number of new polio cases at time t.
Let f(t) be the number of infectives at time t , f(0) = i.
Let C(t) be the number of asymptomatic carriers at time t, C(0) = c.
For $i + c \geq 0$ *and* $i, c \in N \cup \{0\}$, we define the conditional probability function

$X_{i,c} = P\{D(t) = D\{0\}|f(0) = i, C(0) = c\}$
such that
$$x_{i,c}(t) = x_{i,0}(t)^i . x_{0,1}(t)^c \qquad (1)$$

We now impose these conditions on the state of the process in the interval (t, t + Δt] to derive backward equations for $x_{i,c}(t)$ when i = 1, c = 0 or i = 0, c = 1, together with the following the properties of the Poisson process:

(i) $P\{D(t+\Delta t) - D(t) = 1\} = \beta_{i,j}\Delta t + o(\Delta t)$
(ii) $P\{D(t+\Delta t) - D(t) > 1\} = o(\Delta t)$ (2)
(iii) $P\{D(t+\Delta t) - D(t) = 01\} = 1 - \beta_{i,j}(\Delta t)$

Where $\beta_{i,j}$ is the Poisson rate at which type i transmission agent gives rise to type j transmission agent

$$x_{1,0}(t+\Delta t) = \beta_{1,1}\Delta t x_{2,0}(t) + \beta_{1,2}\Delta t x_{1,1}(t) + [1 - \Delta t(\beta_{1,1} + \beta_{1,2} + \gamma)]x_{1,0}(t) + o(\Delta t) \qquad (3)$$

But

$x_{1,1}(t) = x_{1,0}(t)^1 . x_{1,0}(t)^1$
and
$$x_{2,0}(t) = p_{1,0}(t)^2 \qquad (4)$$

Thus,
$$x_{1,0}(t+\Delta t) = \beta_{1,1}\Delta t x_{1,0}(t)^2 + \beta_{1,2}\Delta t x_{1,0}(t)x_{0,1}(t) + x_{1,0}(t) - \Delta t(\beta_{1,1} + \beta_{1,2} + \gamma)x_{1,0}(t) + o(\Delta t) \qquad (5)$$
$$x_{1,0}(t+\Delta t) - x_{1,0}(t) = [\beta_{1,1}x_{1,0}(t)^2 + \beta_{1,2}x_{1,0}(t)x_{0,1}(t) - (\beta_{1,1} + \beta_{1,2} + \gamma)x_{1,0}(t)](\Delta t)$$
$$\frac{x_{1,0}(t+\Delta t) - x_{1,0}(t)}{\Delta t} = \beta_{1,1}x_{1,0}(t)^2 + \beta_{1,2}x_{1,0}(t)x_{0,1}(t) - (\beta_{1,1} + \beta_{1,2} + \gamma)x_{1,0}(t) \qquad (6)$$

As $\Delta t \to 0$,

$$\frac{dx_{1,0}(t)}{dt} = \beta_{1,1}x_{1,0}(t)^2 + \beta_{1,2}x_{1,0}(t)x_{0,1}(t) - (\beta_{1,1} + \beta_{1,2} + \gamma)x_{1,0}(t) \qquad (7)$$

Similarly, if the interval (t, t + Δt] starts with one carrier and no infective in which case i = 0 and c = 1, the backward equation is

$$\frac{dx_{01}(t)}{dt} = \beta_{2,2}x_{0,1}(t)^2 + \beta_{2,1}x_{1,0}(t)x_{0,1}(t) - (\beta_{2,2} + \beta_{2,1} + \mu)x_{0,1}(t) + \mu \qquad (8)$$

Equations (7) and (8) are subject to the conditions $x_{1,0}(0) = x_{0,1}(0) = 1$ because an initial transmission agent must exist with probability 1 else the epidemic process would not be.
We now solve for $x_{1,0}(0)$ *and* $x_{0,1}(0)$ jointly numerically using Picard's method using the representations, x = $x_{0,1}(t)$, y = $x_{1,0}(t)$, $a_1 = \beta_{1,1}$, $b_1 = \beta_{1,2}$, $a_2 = \beta_{2,2}$, $b_2 = \beta_{2,1}$, $d_1 = \beta_{1,1} + \beta_{1,2} - \gamma$ and $d_2 = \beta_{2,,2} + \beta_{2,1} - \mu$

Thus, equations (7) and (8) become

$y' = a_1y^2 + b_1yx - dy$
and
$x' = a_2x^2 + b_2yx - dx + \mu$
subject to
$$y(0) = x(0) = 1 \qquad (9)$$

Now,

$$y = y(0) + \int_0^t y(x,y)dt$$
$$= 1 + \int_0^t (a_1y^2 + b_1xy - d_1y)dt$$
$$= 1 + \int_0^t (a_1 + b_1 - d_1)dt$$
$$= 1 + (a_1 + b_1 - d_1)t$$
$$y = 1 + \xi_1 t$$
where $\xi_1 = a_1 + b_1 - d_1$ (10)
and
$$x = x(0) + \int_0^t x(x,y)dt$$
$$= 1 + \int_0^t (a_2x^2 + b_2xy - d_2x + \mu)dt$$
$$= 1 + \int_0^t (a_2 + b_2 - d_2 + \mu)dt$$
$$= 1 + (a_2 + b_2 - d_2 + \mu)t$$
$$x = 1 + \xi_2 t$$
where $\xi_1 = a_2 + b_2 - d_2 + \mu$ (11)
$$y_2 = 1 + \int_0^t \{a_1(1+\xi_1 t)^2 + b_1(1+\xi_1 t)(1+\xi_2 t) - d_1(1+\xi_1 t)\}dt \qquad (12)$$
$$= 1 + \int_0^t \{a_1 + 2a_1\xi_1 t + \xi_1^2 a_1 t^2 + b_1 + b_1(\xi_1+\xi_2)t + \xi_1\xi_2 t^2 b_1 - d_1 - d_1\xi_1 t\}dt$$
$$= 1 + a_1t + 2a_1\xi_1\frac{t^2}{2} + a_1\xi_1^2\frac{t^3}{3} + b_1t + b_1(\xi_1+\xi_2)\frac{t^2}{2} + b_1\xi_1\xi_2\frac{t^3}{3} - d_1t - d_1\xi_1\frac{t^2}{2}$$

$$y_2 = 1 + (a_1 + b_1 - d_1)t + (2a_1\xi_1 + b_1(\xi_1 + \xi_2) - d_1\xi_1)\frac{t^2}{2} + (a_1\xi_1^2 + \xi_1\xi_2 b_1)\frac{t^3}{3} \quad (13)$$

$$x_2 = 1 + \int_0^t \{a_2(1 + \xi_2 t)^2 + b_2(1 + \xi_1 t)(1 + \xi_2 t) - d_2(1 + \xi_2 t) + \mu\}dt \quad (14)$$

$$= 1 + \int_0^t \{a_2 + 2a_2\xi_2 t + a_2\xi_2^2 t^2 + b_{2+} b_2(\xi_1 + \xi_2)t + \xi_1\xi_2 b_2 t^2 - d_2 - d_2\xi_2 t + \mu\}dt$$

$$= 1 + \int_0^t \{(a_2 + b_2 - d_2 + \mu) + (2a_2\xi_2 + b_2(\xi_1 + \xi_2) - d_2\xi_2)t + (a_2\xi_2^2 + \xi_1\xi_2 b_2)t^2\}dt$$

$$x_2 = 1 + (a_2 + b_2 - d_2 + \mu)t + (2a_2\xi_2 + b_2(\xi_1 + \xi_2) - d_2\xi_2)\frac{t^2}{2} + (a_2\xi_2^2 + \xi_1\xi_2 b_2)\frac{t^3}{3} \quad (15)$$

Now,

$$\xi_1 = a_1 + b_1 - d_1$$
$$= \beta_{1,1} + \beta_{1,2} - \beta_{1,1} - \beta_{1,2} - \gamma = -\gamma \quad (16)$$

and

$$\xi_2 = a_2 + b_2 - d_2 \quad (17)$$
$$= \beta_{2,2} + \beta_{2,1} - \beta_{2,2} - \beta_{2,1} - \mu = -\mu$$

So,

$$x_{1,0}(t) = 1 - \gamma t + [2\beta_{1,1}(-\gamma) + \beta_{1,2}(-\gamma - \mu) + \gamma(\beta_{1,1} + \beta_{1,2} + \gamma)]\frac{t^2}{2} + (\beta_{1,1}\gamma^2 + \mu\gamma\beta_{1,2})\frac{t^3}{3}$$
$$= 1 - \gamma t + (-\gamma\beta_{1,1} - \mu\beta_{1,2} + \beta_{1,2} + \gamma^2)\frac{t^2}{2} + (\beta_{1,1}\gamma^2 + \mu\gamma\beta_{1,2})\frac{t^3}{3} \quad (18)$$

$$X_{1,0}(t) = 1 - \gamma t - (\gamma\beta_{1,1} + \mu\beta_{1,2} - \gamma)^2\frac{t^2}{2} + (\beta_{1,1}\gamma^2 + \mu\gamma\beta_{1,2}\frac{t^3}{3} \quad (19)$$

Also,

$$x_{0,1}(t) = x$$
$$= 1 + (\beta_{2,2} + \beta_{2,1} - \beta_{2,2} - \beta_{2,1} - \mu + \mu)t + [2\beta_{2,2}(-\mu) + \beta_{2,1}(-\gamma - \mu) + \mu(\beta_{2,2} + \beta_{2,1}) + \mu^2]\frac{t^2}{2} + (\beta_{2,2}\mu^2 + \gamma\mu\beta_{2,1})\frac{t^3}{3} \quad (20)$$

$$x_{0,1}(t) = 1 + 0(t) + [-2\beta_{2,2}\mu - \gamma\beta_{2,1} - \mu\beta_{2,1} + \mu\beta_{2,2} + \mu\beta_{2,1} + \mu^2]\frac{t^2}{2} + (\beta_{2,2}\mu^2 + \gamma\mu\beta_{2,1})\frac{t^3}{3}$$
$$= 1 + (-\beta_{2,2}\mu - \gamma\beta_{2,1})\frac{t^2}{2} + (\beta_{2,2}\mu^2 + \gamma\mu\beta_{2,1})\frac{t^3}{3}$$
$$= 1 - (-\beta_{2,2}\mu - \gamma\beta_{2,1} + \mu^2)\frac{t^2}{2} + (\beta_{2,2}\mu^2 + \gamma\mu\beta_{2,1})\frac{t^3}{3}$$

Thus,

$$x_{0,1}(t) = 1 - (\beta_{2,2}\mu - \gamma\beta_{2,1} - \mu^2)\frac{t^2}{2} + (\beta_{2,2}\mu^2 + \gamma\mu\beta_{2,1})\frac{t^3}{3} \quad (21)$$

Corollary

Given that $g_n(s)$ is the probability generating function of X_n and $g_0(s) = s$, then

$$g_{n+1}(s) = \sum_{x\geq 0}(P\{x_{n+1} = x\})s^n \quad (22)$$

CONCLUSION

The extinction probability of the transmission agents, given no new discovered cases in the time interval (s,t] is formulated from the numerical solutions $x_{0,1}(t)$, $x_{1,0}(t)$ together with $q_{0,1}(t)$ of Olawuwo and Ugbebor ([OU18]) as follows,

$$P\{I(t) + C(t) = 0 | D(s) = D(t)\}$$
$$= \frac{P\{f(t) + C(t) = 0 \text{ and } D(s) = D(t)\}}{P\{D(s) = D(t)\}}$$
$$= \frac{\sum_{k=0}^{\infty} q_{0,1}(t-s)^k P\{C(s) = k\}}{\sum_{j,k=0}^{\infty} x_{1,0}x_{1,0}(t-s)^j x_{0,1}(t-s)^k P\{I(s) = j, C(s) = k\}}$$

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