ON SOME PERMUTATION STATISTICS OF THE AUNU PATTERN $\omega_i \in G_p$

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ABSTRACT: This paper examine certain behavior of some permutation statistics on the pattern ω_i defined by Garba and Ibrahim ([GI09]). We investigate the permutation statistics that are equidistributed on $\omega_i \in G_p$ for any prime p≥5. This to a larger extent helped in identifying ω_i as an Eulerian distribution. The graph permutation of ω_i and the graph of 132-avoiding pattern of ω_i were also studied.
KEYWORDS: Permutation Pattern, Permutation Statistics and Graph.

1. INTRODUCTION

Permutation since inception has become a strong mathematical tool for generating various mathematical structures ranging from symmetric groups, pattern avoidance and topological group.
Throughout the extensive history of permutation, the sole aim is the arrangement of objects or numbers. These arrangements could be in a deranged or non-deranged form. A deranged permutation is a permutation with a fixed element or point while a non-deranged permutation is a permutation with no fixed element or point.
The study of permutation pattern date back at least to Knuth ([DE17]). There are permutations that contain certain pattern called pattern containment, and some avoid certain pattern called pattern restriction or forbidding/avoidance.
Several generalizations of patterns have been introduced. The first extension was the vincular pattern defined by Babson and Steingrimsson in 2000 ([BS14]), others include: the bivincular pattern by Bousquet-Me'lou et al ([BUC17]), the mesh pattern by Brande'n and Claesson ([H+15]), and the Barred pattern defined by Pudwell, ([Pud10]).
Garba and Ibrahim ([GI09]), uses the combinatorics approach of the Catalan numbers on a scheme of primes p: $5 \le p \le 11$ and $\Omega \subseteq$ □, to generate the cycles ω_i called the Aunu permutation patterns earlier studied by Ibrahim ([Ibr05]). The Aunu permutation pattern is an interpretation of the Bara'at model ([IEK16]).
In this paper we investigate the permutation statistics on ω_i.

2. DEFINITION AND PRELIMINARY

Definition 1: A *permutation* α on a set X= {1,2,…,n} is the bijection α(i):X→X. it is represented in two line notion as: $\alpha(\text{i})=\begin{pmatrix} 1 & 2 & \cdots & n \\ \alpha(1) & \alpha(2) & \cdots & \alpha(n) \end{pmatrix}$ and $\alpha(\text{i}) = (\alpha(\text{i})\ \alpha(2) \ldots\ \alpha(\text{n}))$ in one line notation. for i=1,2,……n.
The permutation α is said to be *alternating* if $\alpha_1 < \alpha_2 > \alpha_3 < \ldots$ ([GJ13]).
Definition 2: *Aunu pattern* is a non-deranged permutation on the set X= {1, 2,…,p} fixing the first element one, for any prime p≥5. The permutation has a generating function:

$$\omega_i = \left(1\ (1+i)^{modp}\ (1+2i)^{modp}\ \ldots (1 + ((p-1)i)^{modp}\right) \ldots \ldots \ldots (1)$$

for $i = 1,\ 2, ..p-1$.
Definition 3: *Pattern avoidance*: let $q = (q_1 \ldots \ldots q_k) \in S_k$ be a permutation and let k ≤ n. We say that the permutation $\text{t} = (t_1 \ldots \ldots t_n) \in S_n$ is q–avoiding if there is no $1 \le i_{q_1} < i_{q_2} < \cdots < i_{q_k} \le n$ such that $p(i_1) < p(i_2) \ldots . < p(i_k)$.
Definition 4: *Permutation statistics* is a function $S: S_n \to F$, where F is any fixed set. In a simpler form, is a function from S_n to a non-negative number.
Let $\alpha=\alpha_1,\alpha_2,\ldots\ldots\alpha_n \epsilon\ S_n$, then we have;

- The number of *inversions*, $inv(\alpha) = |Inv(\alpha)| = |\{(i,j): i < j \text{ and } \alpha(i) > \alpha(j)\}|$.
- The number of *coinversions*, $coinv(\alpha) = |Coinv(\alpha)| = |\{(i,j): i < j \text{ and } \alpha(i) < \alpha(j)\}|$.
- The number of *descents*, $des(\alpha) = |Des(\alpha)| = |\{i\colon \alpha(i) > \alpha(i+1)\}|$.
- The number of *excedances*, $exc(\alpha) = |Exc(\alpha)| = |\{i\colon \alpha(i) > i\}|$.
- The *major index*, as the sum of the positions of the descent i.e. $maj(\alpha) = \sum_{\alpha(i) > \alpha(i+1)} i$.

Definition 5: A permutation is said to be *Eulerian* if it is equidistributed in its descent and excedance, and *Mahonian* if it is equidistributed in its inversion and major index.

Thus, the statistics inversion and major index are both Mahonian statistics while, descent and excedance are both Eulerian statistics.

Definition 6: A *graph* G is the collection (V, E) of vertices $V_i$edges E= (v_i,v_j). The graph is called a *digraph* if the edge set is made of ordered vertex pair, otherwise it is *undirected.* If any two vertices of G are not connected by a path then G is said to be *connected* while if at least two vertices of G are not connected by a path G is said to be *disconnected.*

Definition 7: A *complete graph* K_n on n vertices is a simple graph with an edge between every pair of vertices. The number of edges in a complete graph is given by |E| =$^nC_2$.

A graph is a called *planar* if it can be drawn on a plane without any edges crossing. Otherwise it is called *Nonplanar*.

3. METHODOLOGY

We cross examine the Aunu pattern ω_i using the classical statistics; inversion, coinversion, descent, major index, and excedance.

We equally employ the theoretical definition of graph, with the restriction

$$Inv(\omega_i) = \{(j,k): j < k \; and \; \omega_i(j) > \omega_i(k)\}$$

on the edges of ω_i graphs, and

$$Coinv(\omega_i) = \{(j,k): j < k \; and \; \omega_i(j) < \omega_i(k)\}$$

on the edges of the Aunu permutations that avoid the pattern 132. The ω_i avoiding 132-pattern was adopted from Mustafa and Ibrahim ([MI13]).

4. RESULTS

Theorem 1: For any p≥5, the permutation pattern $\omega_i \in G_p$ is Eulerian.

Proof: let $des(\omega_i) = |\{j: \omega_i(j) > \omega_i(j+1)\}|$ and $exc(\omega_i) = |\{j: \omega_i(j) > j\}|$

By definition, it implies that for each $des(\omega_i)$ and $exc(\omega_i)$, there exist a unique value c such that;

$$\sum_{i=1}^{p-1} des\omega_i = \sum_{i=1}^{p-1} exc\omega_i = c.$$

Proposition 2: let $|G_p|$ be the cardinality of G_p. If ω_{iA} is the number of ω_i with alternating pattern. Then for any $p \geq 5$, and $G_p = \{\omega_1 \dots \omega_{p-1}\}$.

$$\omega_{iA} = p - |G_p|.$$

Proof: If $G_p = \{\omega_1 \dots \omega_{p-1}\}$ and given ω_i by (1). Then, clearly $|G_p| = p - 1$. Thus, there exist a ω_i such that;

$$\omega_{iA} = p - |G_p|.$$

Remark: There is only one $\omega_i \in G_p$ for any $p \geq 5$ that alternate.

Proposition 3: let ω_{iA} be an alternating pattern and ω_{iA}^r the reverse permutation of ω_{iA}. Then the major index enumeration of ω_{iA} is the same as the major index enumeration of ω_{iA}^r.

Proof: let $\omega_{iA} = (\alpha_1 < \alpha_2 > \cdots .)$ and $\omega_{iA}^r = (\alpha_5 > \alpha_4 < \alpha_3 >. \dots .)$.

By definition, we note that $des(\omega_{iA})$ and $des(\omega_{iA}^r)$ has the same enumeration. Thus, there exist a unique value x such that;

$$maj(\omega_{iA}) = maj(\omega_{iA}^r) = x.$$

Theorem 4: let ω_{iA} be an alternating permutation of length n. Then, the statistic major index of ω_{iA} and ω_{iA}^r have equal enumeration.

Proof: the proof is trivial from the proof of proposition 3 above.

Theorem 5: let G(V,E) be a graph and $E = Inv\omega_i$ for $\omega_i \in (1,..,p-1)$ and prime $p \geq 5$, then the collection $G = (G_i,\dots,G_{p-1})$ contain $p-2$ graphs.

Proof: let $G_i = (V_i, E_i)$ *so that* $E_i = Inv(\omega_i)$.

Clearly there exist no such $Inv(\omega_i) = \{(j,k): j < k \; and \; \omega_i(j) > \omega_i(k)\}$ for $i = 1$ but true for all $i > 1$. Therefore, G_1 is not a graph. Thus, the cardinality:

$$|G| = p - 2.$$

Remark: The graph G(V,E) is connected.

Proposition 6: The number of edges |E| of the graph G $= (V_i, Coinv(\omega_i))$ for all $\omega_i \in G_p$ and prime $p \geq 5$, avoiding the pattern 132 is giving as;

$$|E| = p(p-1)/2.$$

Proof: The result is trivial by the completeness of a graph and definition of coinversion as above.

Remark: |E|= $= (V_i, Coinv(\omega_i))$ for $\omega_i \in G_p$ and prime $p \geq 5$. Thus G $= (V_i, Coinv(\omega_i))$ is a complete graph.

Proposition 7: The degree $\delta(v_i)$ for each v_i of the graph G $= (V_i, Coinv(\omega_i))$ of all $\omega_i \in G_p$ and prime $p \geq 5$, avoiding the pattern 132 is $p - 1$.

Proof: By definition, edge (E) =(v_i, v_j) where v_i, v_j are pair of vertices.

$$\Rightarrow Coinv(\omega_i) = \{(v_j, v_k): v_j < v_k \; and \omega_i(v_j) < \omega_i(v_k)\}.$$

$\Rightarrow$ For all $e_i \in E$ there exist $^pC_2 - 6 = p - 1$ edges related to each v_i.Thus, $\delta(v_i) = p - 1$.

Remark: The graph $G = (V_i, Coinv(\omega_i))$ for all $\omega_i \in G_p$ and prime $p \geq 5$, avoiding the pattern 132 is Non-planar.

5. CONCLUSION

In this work, we have shown that the pattern $\omega_i \in G_p$ called the Aunu permutation pattern is a Eulerian distribution and that it contain $p - |G_p|$ number of ω_i that has an alternating pattern. We observe that, G_p contain $p - 2$ graphs with the restriction $E = Inv\omega_i$ on the graphs $G = (V_i, Inv(\omega_i))$ and note a relationship between an alternating permutation pattern and it reverse.

We also justify our claim on the graphs $G = (V_i, Coinv(\omega_i))$ graphs that avoids the 132 pattern.

Finally, we discover that the permutation graphs $G = (V_i, Inv(\omega_i))$ and $G = (V_i, Coinv(\omega_i))$ are connected and complete non-planar graphs respectively.

We therefore recommend for further study of the permutation graph of the structure ω_i.

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