DISCOVERING THEOREMS ABOUT THE GAUSSIAN MERSENNE SEQUENCE WITH THE MAPLE´S HELP

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ABSTRACT: In Brazil, we are interested in the study of second-order recurrent numerical sequences. Such mathematical content has an enormous potential to provide a differentiated mathematical culture for the math teachers in initial training. However, it becomes important that the teacher in Brazil knows properties beyond the traditional and classic numerical sequences, such as the Fibonacci sequence, Lucas sequence, Pell sequence, Jacobsthal sequence, Narayana sequence, etc. Thus, in the present work, we present a proposal of CAS Maple as technological resources in the study and discovery of new theorems derived from the Mersenne sequence. Throughout the work, we show that the software allows the discovery of theorems from matrix properties derived from the Mersenne sequences and the sequence of Gaussian numbers of Mersenne. The research presented in the paper can stimulate the formation of Mathematics teachers in Brazil.

KEY WORDS: Mersenne sequence, Gaussian sequence of Mersenne, Theorems, Teacher training, Technology.

1. INTRODUCTION

In the context of the formation of Mathematics teachers in Brazil, we sometimes observe a tendency of authors of Mathematical History books to dedicate an exagerated attention only to the bases of differential and integral calculus, while other mathematical contents, from the point of view of their historical origin and emergence, remain disregarded by the books of History of Mathematics published and used in Brazilian universities. ([AA18]).

On the other hand, an important aspect related to the teaching of Mathematics in the historical, mathematical and evolutionary context concerns the possibilities of using the technology and its possibilities, with the interest of providing Mathematics teachers in Brazil a differentiated itinerary for direct contact both with Mathematics, as well as the perception of the potentialities of the current Technology and the implementation of a scientific research itinerary with an extended set of theoretical and conceptual tools.

From these preliminary considerations, we consider the Mersenne sequence as an object of discussion and study. We consider the following set indicating some of these values $(0, 1, 3, 7, 15, 31, 63, 127, \dots, M(n))$.

The Mersenne numbers are defined recurrently by the inhomogeneous equation $M(n+1) = 2M(n) + 1$. On the other hand, we can write $M(n+2) = 2M(n+1) + 1$. From these two equations, we can easily get a homogeneous equation ([CCV16]).

In fact, we write $M(n+2) - M(n+1) = 2M(n+1) - 2M(n) + 1) = 2M(n+1) - 2M(n)$. Finally, we have $M(n+2) = 3M(n+1) - 2M(n)$. From this, we take the following mathematical definition for the Mersenne sequence.

Definition 1: The Mersenne sequence is defined by the recurrently relation indicated $M(n+2) = 3M(n+1) - 2M(n)$, with initial conditions $M(0) = 0, M(1) = 1, n \geq 0$.

Moreover, we can determine the elements of the set $(M(-n))$. Let's look at some of them in the list below, for every integer $n \geq 0$:

$$\left(\dots, M_{-n}, \dots, -\frac{7}{8}, -\frac{3}{4}, -\frac{1}{2}, 0,1,3,7, \dots, M_n, \dots\right)$$

However, directly from the fundamental recurrence relation $M_{n+2} = 3M_{n+1} - 2M_n, n \geq 0$ we can determine such elements indicated by $\langle M_{-n}\rangle_{n\in\mathbb{N}}$, considering now the set of the negative integer indices.

We can also consider that $M_{n+2} = 3M_{n+1} - 2M_n \leftrightarrow$

$$\leftrightarrow \frac{M_{n+2}}{M_{n+1}} = 3 - 2\frac{M_n}{M_{n+1}} \leftrightarrow \frac{M_{n+2}}{M_{n+1}} = 3 - 2\frac{1}{\left(\frac{M_{n+1}}{M_n}\right)}$$

We will take $\frac{M_{n+2}}{M_n} = t_{n+1}$ and replace it in the last

equation $t_{n+2} = 3 - 2.\frac{1}{t_{n+1}}$. Finally, we consider that exists the following limit $\lim_{n\to\infty} t_n = t \in \mathbb{R}$ and we can take the identity $t_{n+2}.t_{n+1} = 3t_{n+1} - 2$. If we take the symbol of limit in the both side, we will find that $t^2 = \lim_{n\to\infty}(t_{n+2}\, t_{n+1}) = \lim_{n\to\infty}(3t_{n+1} - 2) = 3t - 2 \leftrightarrow t^2 = 3t - 2$. From this equation, we can verify that $t^2 = 3t - 2 = (M_2.t - 2.M_1)$. Moreover, we can still find that $t^3 = t.t^2 = (3t-2)t = 3t^2 - 2t = 3(3t-2) - 2t = 7t - 2.3 = M_3.t - 2.M_2$. We can easily determine the roots of the quadratic equation $t^1 - 3t + 2 = 0$, with the real roots indicated by $r_1 = 1, r_2 = 2$.
By mathematical induction, we can find that $r_1^n = M_n.r_1 - 2.M_{n-1}$ and $r_2^n = M_n.r_2 - 2.M_{n-1}$, for every integer $n \geq 0$. Finally, we will enunciate the first theorem.

Theorem 1: For every integer $n \geq 0$, we have: $(i)\ M_n = 2^n - 1$; $(ii)\ M_{-n} = -\frac{M_n}{2^n}$.

Proof. We have already determined that $r_1^n = M_n.r_1 - 2.M_{n-1}$ and $r_2^n = M_n.r_2 - 2.M_{n-1}$. We can consider the following difference $r_1^n - r_2^n = (M_n.r_1 - 2.M_{n-1}) - (M_n.r_2 - 2.M_{n-1}) = M_n(r_1 - r_2)$. Finally, we find that $M_n = \frac{r_1^n - r_2^n}{r_1 - r_2}$, where $r_1 = 1, r_2 = 2$. In this way, we can also write that $M_n = \frac{1^n - 2^n}{1-2} = 2^n - 1$. On the other hand, we can replace the index $'n'$ by $'-n'$ and determine another formula $M_{-n} = (2^{-n} - 1) = \left(\frac{1}{2}\right)^n - 1 = -\frac{1}{2^n}(2^n - 1) = \left(-\frac{M_n}{2^n}\right)$, for every integer $n \geq 0$. □

From the work ([CCV16]), we can consider the following matrix $IM = \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix}$. In the subsequent section we will develop the study of several numerical and mathematical properties derived from the matrix power of the special type $IM^n = \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix}^n$ and $IM^{-n} = \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix}^{-n}$, where $n \geq 0$.
In the first case, for example, we can determine that:

$$IM = \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix} = \begin{pmatrix} -2M_0 & M_1 \\ -2M_1 & M_2 \end{pmatrix},$$
$$IM^2 = \begin{pmatrix} -2 & 3 \\ -6 & 7 \end{pmatrix} = \begin{pmatrix} -2M_1 & M_2 \\ -2M_2 & M_3 \end{pmatrix},$$
$$IM^3 = \begin{pmatrix} -6 & 7 \\ -14 & 15 \end{pmatrix} = \begin{pmatrix} -2M_2 & M_3 \\ -2M_3 & M_4 \end{pmatrix},$$
$$IM^4 = \begin{pmatrix} -14 & 15 \\ -30 & 31 \end{pmatrix} = \begin{pmatrix} -2M_3 & M_4 \\ -2M_4 & M_5 \end{pmatrix},$$
$$IM^5 = \begin{pmatrix} -30 & 31 \\ -62 & 63 \end{pmatrix} = \begin{pmatrix} -2M_4 & M_5 \\ -2M_5 & M_6 \end{pmatrix},$$
$$IM^6 = \begin{pmatrix} -62 & 63 \\ -126 & 127 \end{pmatrix} = \begin{pmatrix} -2M_5 & M_6 \\ -2M_6 & M_7 \end{pmatrix},$$
$$IM^7 = \begin{pmatrix} -126 & 127 \\ -254 & 255 \end{pmatrix} = \begin{pmatrix} -2M_5 & M_7 \\ -2M_7 & M_8 \end{pmatrix},$$
$$IM^8 = \begin{pmatrix} -254 & 255 \\ -510 & 511 \end{pmatrix} = \begin{pmatrix} -2M_7 & M_8 \\ -2M_8 & M_9 \end{pmatrix},$$
$$IM^9 = \begin{pmatrix} -510 & 511 \\ -1022 & 1023 \end{pmatrix} = \begin{pmatrix} -2M_8 & M_9 \\ -2M_9 & M_{10} \end{pmatrix},$$
$$IM^{10} = \begin{pmatrix} -1022 & 1023 \\ -2046 & 2047 \end{pmatrix} = \begin{pmatrix} -2M_{n-1} & M_n \\ -2M_n & M_{n+1} \end{pmatrix},$$
etc.

For the matrix case $IM^{-n} = \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix}^{-n}$ we can verify that:

$$IM^{-1} = \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix}^{-1} = \begin{pmatrix} -\frac{3}{2} & -\frac{1}{2} \\ 1 & 0 \end{pmatrix} =$$
$$\begin{pmatrix} -2\left(-\frac{3}{4}\right) & -\frac{1}{2} \\ -2\left(-\frac{1}{2}\right) & 0 \end{pmatrix} = \begin{pmatrix} -2M_{-2} & M_{-1} \\ -2M_{-1} & M_0 \end{pmatrix}, IM^{-2} =$$
$$\begin{pmatrix} \frac{7}{4} & -\frac{3}{4} \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix} = \begin{pmatrix} -2M_{-3} & M_{-2} \\ -2M_{-2} & M_{-1} \end{pmatrix} = M_{-2},\ IM^{-3} =$$
$$\begin{pmatrix} \frac{15}{8} & -\frac{7}{8} \\ \frac{7}{4} & -\frac{3}{4} \end{pmatrix} = \begin{pmatrix} -2M_{-4} & M_{-3} \\ -2M_{-3} & M_{-2} \end{pmatrix} = M_{-3},\ IM^{-4} =$$
$$\begin{pmatrix} \frac{31}{16} & -\frac{15}{16} \\ \frac{15}{8} & -\frac{7}{8} \end{pmatrix} = \begin{pmatrix} -2M_{-5} & M_{-4} \\ -2M_{-4} & M_{-3} \end{pmatrix} = M_{-4}, IM^{-5} =$$
$$\begin{pmatrix} \frac{63}{32} & -\frac{31}{32} \\ \frac{31}{16} & -\frac{15}{16} \end{pmatrix} = \begin{pmatrix} -2M_{-6} & M_{-5} \\ -2M_{-5} & M_{-4} \end{pmatrix} = M_{-5}, IM^{-6} =$$
$$\begin{pmatrix} \frac{127}{64} & -\frac{63}{64} \\ \frac{63}{32} & -\frac{31}{12} \end{pmatrix} = \begin{pmatrix} -2M_{-7} & M_{-6} \\ -2M_{-6} & M_{-5} \end{pmatrix} = M_{-6}, IM^{-7} =$$
$$\begin{pmatrix} \frac{255}{128} & -\frac{127}{128} \\ \frac{127}{64} & -\frac{63}{64} \end{pmatrix} = \begin{pmatrix} -2M_{-8} & M_{-7} \\ -2M_{-7} & M_{-6} \end{pmatrix} = M_{-7}, IM^{-8} =$$
$$\begin{pmatrix} \frac{511}{128} & -\frac{255}{256} \\ \frac{255}{128} & -\frac{127}{128} \end{pmatrix} = \begin{pmatrix} -2M_{-9} & M_{-8} \\ -2M_{-8} & M_{-7} \end{pmatrix} = M_{-8},$$
M_{-9}, M_{-10}, etc.

We have observed that these particular examples of matrices powers are determined with the computational resource. Thus, before proceeding and indicating how CAS Maple can help in our mathematical and improve our scientific investigative process. With this purpose, we will define the following special matrices from definition 2. We will verify with CAS Maple that the Mersenne numbers can be generated from these matrices, from the expanded set of data enabled by the software. Many of these matriarchal properties become quite tiresome when we disregard current technology for the teaching and learning of

Mathematics and the History of Mathematics. ([Alv18]).

Definition 2: For every positive integer $n \geq 0$, the matrices determined by the Mersenne numbers are defined by two conditions:

$(i)\ IM_n = \begin{pmatrix} -2M_{n-1} & M_n \\ -2M_n & M_{n+1} \end{pmatrix}$, for every integer $n \geq 0$;

$(ii)\ IM_{-n} = \begin{pmatrix} -2M_{-n-1} & M_{-n} \\ -2M_{-n} & M_{-n+1} \end{pmatrix} = \begin{pmatrix} -2M_{-(n+1)} & M_{-n} \\ -2M_{-n} & M_{-(n-1)} \end{pmatrix}$, for every integer $n \geq 0$.

In the next section we will see a research itinerary and how the CAS Maple can help us to identify certain properties related to the Mersenne sequence.

2. DISCOVERING THEOREMS ABOUT THE MERSENNE SEQUENCE

Now, we will explore some matrix representations related to the Mersenne sequence. We will verify that when we deal with the Mersenne sequence the operational calculation becomes quite complicated and the use of software such as Maple can provide the exploration of an investigative process based in the inductive thinking aiming the confirmation of certain important mathematical and combinatorial properties. In a preliminary way, we present some particular cases of the Mersenne sequence.

With the computational resource, we will show that matrices of type $(IM^{-1})^n = IM^{-n} = \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix}^{-n}$ determine the elements of the Mersenne sequence with negative indices. Indeed, from the previous examples, we have determined, for example, that

$$IM^{-5} = \begin{pmatrix} \frac{63}{32} & -\frac{31}{32} \\ \frac{31}{16} & -\frac{15}{16} \end{pmatrix} = \begin{pmatrix} -2M_{-6} & M_{-5} \\ -2M_{-5} & M_{-4} \end{pmatrix} = IM_{-5}.$$

For example, in Figure 1, we show determination of the matrix

$$(IM^{-1})^{20} = IM^{-20} = \begin{pmatrix} -2M_{-8} & M_{-7} \\ -2M_{-7} & M_{-6} \end{pmatrix} = IM_{-20}.$$

Another property and theorem that can be discovered with the aid of CAS Maple, for example, refers to the character of commutativity expressed as follows $IM_{n+m} = (IM_n.IM_m) = (IM_m.IM_n) = IM_{m+n}$, for every positive integers $m, n \geq 1$. From commutative properties involving the inverse matrices verified with the software related to inverse matrices, we can formulate mathematical conjectures such as $IM_{-(n+m)} = (IM_{-n}.IM_{-m}) = (IM_{-m}.IM_{-n}) = IM_{-(m+n)}$, for every positive integers $m, n \geq 1$.

From the various particular cases indicated above, we will state the following provisional conjecture.

Conjecture 1: For every integer $n \geq 0$, we have:

$(i)\ IM_n = \begin{pmatrix} -2M_{n-1} & M_n \\ -2M_n & M_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix}^n$;

$(ii)\ IM_{-n} = \begin{pmatrix} -2M_{-n-1} & M_{-n} \\ -2M_{-n} & M_{-n+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix}^{-n} = \left[\begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix}^{-1}\right]^n$.

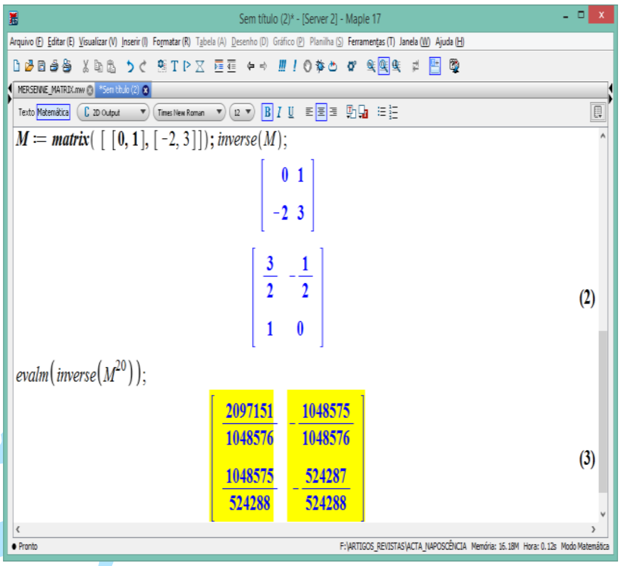

Figure 1. With the help of the CAS Maple we can determine high potencies of matrices related to the Mersenne sequence (Prepared by the author)

The investigative discovery process and theorem verification can be improved with CAS Maple. In this case, we can prove, a posteriori, the veracity of the previous conjecture through the Mathematical Induction model. Immediately, by mathematical induction, we will consider that

$$IM^n = \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix}^n = \begin{pmatrix} -2M_{n-1} & M_n \\ -2M_n & M_{n+1} \end{pmatrix} = (IM_n).$$

Thus, we can directly verify that

$$IM^{n+1} = \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix}^{n+1} = \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix}^n \cdot \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix} = \begin{pmatrix} -2M_{n-1} & M_n \\ -2M_n & M_{n+1} \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix} = \begin{pmatrix} -2M_n & 3M_n - 2M_{n-1} \\ -2M_{n+1} & 3M_{n+1} - 2M_n \end{pmatrix} = \begin{pmatrix} -2M_n & M_{n+1} \\ -2M_{n+1} & M_{n+2} \end{pmatrix} = (IM_{n+1}),$$

since we know that $M_{n+2} = 3M_{n+1} - 2M_n$, from definition 1. For the case of inverse matrices, we know the following relationship $M_{-n} = -\frac{M_n}{2^n}$. We will replace in the special matrix

$$IM_{-n} = \begin{pmatrix} -2M_{-n-1} & M_{-n} \\ -2M_{-n} & M_{-n+1} \end{pmatrix} = \begin{pmatrix} -2\left(-\frac{M_{n+1}}{2^{n+1}}\right) & -\frac{M_n}{2^n} \\ -2\left(-\frac{M_n}{2^n}\right) & -\frac{M_{n-1}}{2^{n-1}} \end{pmatrix} = \begin{pmatrix} \frac{M_{n+1}}{2^n} & -\frac{M_n}{2^n} \\ \frac{M_n}{2^{n-1}} & -\frac{M_{n-1}}{2^{n-1}} \end{pmatrix} = \frac{1}{2^n}\begin{pmatrix} M_{n+1} & -M_n \\ 2M_n & -2M_{n-1} \end{pmatrix} = (IM_n)^{-1}.$$

Finally, we can verify that

$$IM_{-n} = {M_n}^{-1} = \left(\begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix}^n\right)^{-1} = \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix}^{-n}.$$

Thus, with computational resources, we can finally enunciate the discovery of the following theorem related to the Mersenne sequence with the Maple´s help. We note that all properties can be verified for a broad data set provided by the CAS Maple. Thus, we can formulate the following theorem derived from some mathematical invariant elements identified in the preceding information (see figure 1).

Theorem 2: For every integer $n, m \geq 0$, we have:

$(i)\ IM_1^n = \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix}^n = \begin{pmatrix} -2M_{n-1} & M_n \\ -2M_n & M_{n+1} \end{pmatrix} = IM_n;$

$(ii)\ IM_{-n} = (IM_n)^{-1} = \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix}^{-n};$

$(iii)\ IM_{(n+m)} = IM_n.IM_m = IM_m.IM_n;$

$(iv)\ IM_{-(n+m)} = IM_{-n}.IM_{-m} = IM_{-m}.IM_{-n}.$

Proof. Just consider the above arguments.

In the following section we will present and define a second-order recurrent sequence with the introduction of an imaginary unit with the classical property $i^2 = -1$.

3. DISCOVERING THEOREMS ABOUT THE GAUSSIAN MERSENNE SEQUENCE

In this section we will introduce a new mathematical definition.

Definition 3: The Mersenne sequence is defined by the recurrently relation $GM_{n+2} = 3.GM_{n+1} - 2.GM_n$, wtth initial conditions $GM_0 = 0 + i, GM_1 = 1 + 3i$, for every integer $n \geq 0$.

From this new definition, we can determine that: $GM_2 = 3 + 7i = M_2 + M_3 i$, $GM_3 = 7 + 15i = M_3 + M_4 i, GM_4 = 15 + 31i = M_4 + M_5 i, GM_5 = 31 + 63i = M_5 + M_6 i, GM_7 = 127 + 255i = M_7 + M_8 i$, etc. From these relations, we will see some arithmetical properties for the Gaussian numbers of Mersenne.

Let us now see some examples of the determination of the Gaussian numbers of Mersenne, for negative integer indices. In fact, we can see that if we take

$$GM_{-1} = \frac{3GM_0 - GM_1}{2} = \frac{3i - 1 - 3i}{2} = -\frac{1}{2} = \frac{1 + 2i - 2 - 2i}{2} = \frac{1 + 2i}{2^1} - (1 + i).$$

We can see another example if we take

$$GM_{-2} = \frac{3GM_{-1} - GM_0}{2} = -\frac{3}{4} - \frac{1}{2}i = \frac{-3 - 2i}{4} = \frac{1 + 2i - 4 - 4i}{4} = \frac{1 + 2i}{2^2} - (1 + i).$$

We can also verify that:

$GM_{-3} = -\frac{7}{8} - \frac{3}{4}i = \frac{-7-6i}{8} = \frac{1+2i-8-8i}{4} = \frac{1+2i}{2^3} - (1 + i), GM_{-4} = -\frac{15}{16} - \frac{7}{8}i = \frac{1+2i}{2^4} - (1 + i),$ $GM_{-5} = -\frac{31}{32} - \frac{15}{16}i = \frac{1+2i}{2^5} - (1 + i),\ GM_{-6} = -\frac{63}{64} - \frac{31}{32}i = \frac{1+2i}{2^6} - (1 + i)$, etc.

Now, we will explore some matrix representations related to the Gaussian Mersenne sequence. We will verify that when we deal with the Gaussian Mersenne sequence the operational calculation becomes quite complicated and the use of software such as CAS Maple can provide the exploration of an investigative process based on the inductive thinking aiming at the confirmation of certain important mathematical properties. In a preliminary way, we present some particular cases of the Gaussian Mersenne sequence denoted by $\langle GM_n \rangle_{n \in \mathbb{N}}$. For this, we define the following matrix $IN = \begin{pmatrix} 1 & i \\ -2i & 1 + 3i \end{pmatrix}$. Next, we will use the following formal definition

Definition 4: For every positive integer $n \geq 0$, the matrices determined by the Gaussian Mersenne numbers are defined by two conditions:

$(i)\ GIM_n = \begin{pmatrix} -2GM_{n-1} & GM_n \\ -2GM_n & GM_{n+1} \end{pmatrix}$, for every integer $n \geq 0$;

$(ii)\ GIM_{-n} = \begin{pmatrix} -2GM_{-n-1} & GM_{-n} \\ -2GM_{-n} & GM_{-n+1} \end{pmatrix} = \begin{pmatrix} -2GM_{-(n+1)} & GM_{-n} \\ -2GM_{-n} & GM_{-(n-1)} \end{pmatrix}$, for every integer $n \geq 0$.

Thus, for every integer $n \geq 0$, we will investigate the behavior of the following matrices indicated by $\begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix}^n \begin{pmatrix} 1 & i \\ -2i & 1 + 3i \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix}^{-n} \begin{pmatrix} 1 & i \\ -2i & 1 + 3i \end{pmatrix}$. Indeed, we can determine with the help of CAS Maple, which we have the matrix property

$$(IM.IN) = (IN.IM) = \begin{pmatrix} -2.1i & 1+3i \\ -2(1+3i) & 3+7i \end{pmatrix},$$

$$IM^2.IN = IN.IM^2 = \begin{pmatrix} -2(1+3i) & 3+7i \\ -2(3+7i) & 7+15i \end{pmatrix},$$

$$IM^3.IN = IN.IM^3 = \begin{pmatrix} -2(3+7i) & 7+15i \\ -2(7+15i) & 15+31i \end{pmatrix},$$

$$IM^4.IN = IN.IM^4 = \begin{pmatrix} -2(7+15i) & 15+31i \\ -2(15+31i) & 31+63i \end{pmatrix},$$

$$IM^5.IN = IN.IM^5 = \begin{pmatrix} -2(15+31i) & 31+63i \\ -2(31+63i) & 63+127i \end{pmatrix},$$

$$IM^6.IN = IN.IM^6 = \begin{pmatrix} -2(31+63i) & 63+127i \\ -2(63+127i) & 127+255i \end{pmatrix},$$

$$IM^7.IN = IN.IM^7 = \begin{pmatrix} -2(63+127i) & 127+255i \\ -2(127+255i) & 255+511i \end{pmatrix},$$

$$IM^8.IN = IN.IM^8 = \begin{pmatrix} -2(127+255i) & 255+511i \\ -2(255+511i) & 511+1023i \end{pmatrix}, \text{ etc.}$$

We exemplify that with CAS Maple we can determine matrices with high powers of the type $IM^{20}.IN = IN.IM^{20}$ or $IM^{30}.IN = IN.IM^{30}$.
We can also verify the commutativity of all these matrix products. In Figure 2 we can see the determination of matrix powers involving the new Gaussian numbers of Mersenne $\langle GM_n \rangle_{n\in\mathbb{N}}$ that we introduced in this work.
The heuristic, inductive and investigative process, from the behavior of all these cases, allows the preliminary discovery or conjecture of the following theorem involving the Gaussian numbers of Mersenne. Thus, we can formulate the following conjecture derived from the invariant elements identified in the preceding information.
Conjecture 2: For every integer $n \geq 0$, we have:
(i) $GIM_n = \begin{pmatrix} -2GM_{n-1} & GM_n \\ -2GM_n & GM_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix}^n \begin{pmatrix} 1 & i \\ -2i & 1+3i \end{pmatrix} = \begin{pmatrix} 1 & i \\ -2i & 1+3i \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix}^n.$

Proof. Considering the inductive step, with the following power matrix
$\begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix}^n \begin{pmatrix} 1 & i \\ -2i & 1+3i \end{pmatrix} = GIM_n = \begin{pmatrix} -2GM_{n-1} & GM_n \\ -2GM_n & GM_{n+1} \end{pmatrix}.$
Let's consider the following matrix product

$$\begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix}^{n+1} \begin{pmatrix} 1 & i \\ -2i & 1+3i \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix}^n \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} 1 & i \\ -2i & 1+3i \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix}^n \begin{pmatrix} 1 & i \\ -2i & 1+3i \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix} =$$
$$\begin{pmatrix} -2GM_{n-1} & GM_n \\ -2GM_n & GM_{n+1} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix} = \begin{pmatrix} -2GM_n & 3GM_n - 2GM_{n-1} \\ -2GM_{n+1} & 3GM_{n+1} - 2GM_n \end{pmatrix} = \begin{pmatrix} -2GM_n & GM_{n+1} \\ -2GM_{n+1} & GM_{n+2} \end{pmatrix} = GIM_{n+1},$$

for every positive integer $n \geq 0$.
On the other hand, we can see directly that:

$$GIM_n = \begin{pmatrix} -2GM_{n-1} & GM_n \\ -2GM_n & GM_{n+1} \end{pmatrix} = \begin{pmatrix} -2(M_{n-1} + M_n i) & M_n + M_{n+1} i \\ -2(M_n + M_{n+1} i) & M_{n+1} + M_{n+2} i \end{pmatrix} = \begin{pmatrix} -2M_{n-1} & M_n \\ -2M_n & M_{n+1} \end{pmatrix} + \begin{pmatrix} -2M_n & M_{n+1} \\ -2M_{n+1} & M_{n+2} \end{pmatrix}.i = \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix}^n +$$
$$\begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix}^{n+1}.i = \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix}^n \times \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & i \\ -2i & 3i \end{pmatrix}\right] = \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix}^n \begin{pmatrix} 1 & i \\ -2i & 1+3i \end{pmatrix}.$$

In figure 2 we see the behavior of matrices powers that generate the Gaussian numbers of Mersenne.

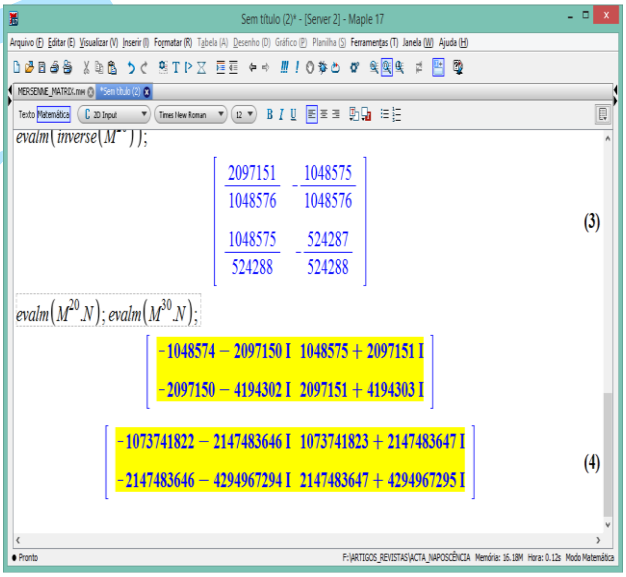

Figure 2. With the help of the CAS Maple we can determine high potencies of matrices related to the Gaussian Mersenne sequence (Prepared by the authors)

From the previous theorem that we established with the aid of the analysis of particular cases, we verified that the matrix determined by the Gaussian numbers of samples completely determined by the matrix generating the numbers of Mersenne $\begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix}^n$ and the multiplication by a fixed matrix indicated by $\begin{pmatrix} 1 & i \\ -2i & 1+3i \end{pmatrix}$.

Let us now see the particular behavior of the following matrix product $IM_1^{-n}.IN$:

$$IM_1^{-1}.IN=IN.IM_1^{-1}=\begin{pmatrix}\frac{3}{2}+i & -\frac{1}{2}+0i\\ 1+0i & 0+1i\end{pmatrix}=\begin{pmatrix}-2\left(-\frac{3}{4}-\frac{1}{2}i\right) & -\frac{1}{2}\\ -2\left(-\frac{1}{2}\right) & 0+i\end{pmatrix}=\begin{pmatrix}-2GM_{-2} & GM_{-1}\\ -2GM_{-1} & GM_0\end{pmatrix}=GIM_{-1},$$

$$IM_1^{-2}.IN=IN.IM_1^{-2}=\begin{pmatrix}\frac{7}{4}+\frac{3}{2}i & -\frac{3}{4}-\frac{1}{2}i\\ \frac{3}{2}+i & -\frac{1}{2}+i\end{pmatrix}=\begin{pmatrix}-2\left(-\frac{7}{8}-\frac{3}{4}i\right) & -\frac{3}{4}-\frac{1}{2}i\\ -2\left(-\frac{3}{4}-\frac{1}{2}i\right) & -\frac{1}{2}\end{pmatrix}=\begin{pmatrix}-2GM_{-3} & GM_{-2}\\ -2GM_{-2} & GM_{-1}\end{pmatrix}=GIM_{-2},$$

$$IM_1^{-3}.IN=IN.IM_1^{-3}=\begin{pmatrix}\frac{15}{8}+\frac{7}{4}i & -\frac{7}{8}-\frac{3}{4}i\\ \frac{7}{4}+\frac{3}{2}i & -\frac{3}{4}-\frac{1}{2}i\end{pmatrix}=\begin{pmatrix}-2\left(-\frac{15}{16}-\frac{7}{8}i\right) & -\frac{7}{8}-\frac{3}{4}i\\ -2\left(-\frac{7}{8}-\frac{3}{4}i\right) & -\frac{3}{4}-\frac{1}{2}i\end{pmatrix}=\begin{pmatrix}-2GM_{-4} & GM_{-3}\\ -2GM_{-3} & GM_{-2}\end{pmatrix}=GIM_{-3},$$

$$IM_1^{-4}.IN=IN.IM_1^{-4}=\begin{pmatrix}\frac{31}{16}+\frac{15}{8}i & -\frac{15}{16}-\frac{7}{8}i\\ \frac{15}{8}+\frac{7}{4}i & -\frac{7}{8}-\frac{3}{4}i\end{pmatrix}=\begin{pmatrix}-2\left(-\frac{31}{32}-\frac{15}{16}i\right) & -\frac{15}{16}-\frac{7}{8}i\\ -2\left(-\frac{15}{16}-\frac{7}{8}i\right) & -\frac{7}{8}-\frac{3}{4}i\end{pmatrix}=\begin{pmatrix}-2GM_{-5} & GM_{-4}\\ -2GM_{-4} & GM_{-3}\end{pmatrix}=GIM_{-4}.$$

We can also verify the commutativity of all these matrix products. In Figure 3 we can see the determination of inverse matrix powers involving the new Gaussian numbers of Mersenne $\langle GM_{-n}\rangle_{n\in\mathbb{N}}$ that we introduced in this work.

In Figures 3 and 4 we provide an extended set of data obtained with the computational resource and that can aid in the understanding of the inductive model of recurrence and that will allow the determination and discovery of a new theorem on the Gaussian numbers of Mersenne. In figure 5 we show the behavior of inverse matrices with high exponents and that makes the calculation impracticable when we disregard the technology in the investigative process.

Conjecture 3: For every integer $n \geq 0$, we have:

$$(i)GIM_{-n}=\begin{pmatrix}-2GM_{-n-1} & GM_{-n}\\ -2GM_{-n} & GM_{-n+1}\end{pmatrix}=\begin{pmatrix}0 & 1\\ -2 & 3\end{pmatrix}^{-n}\begin{pmatrix}1 & i\\ -2i & 1+3i\end{pmatrix},\ \text{for } n\geq 0.$$

Proof. From the definition, we have

$$GIM_{-n}=\begin{pmatrix}-2GM_{-n-1} & GM_{-n}\\ -2GM_{-n} & GM_{-n+1}\end{pmatrix}=\begin{pmatrix}-2GM_{-(n+1)} & GM_{-n}\\ -2GM_{-n} & GM_{-(n-1)}\end{pmatrix}$$

$$=\begin{pmatrix}-2\left(-\frac{1}{2^{n+1}}M_{n+1}+2M_ni\right) & -\frac{1}{2^n}(M_n+2M_{n-1}i)\\ -2\left(-\frac{1}{2^{n-1}}M_n+2M_{n-1}i\right) & -\frac{1}{2^{n-1}}(M_{n-1}+2M_{n-2}i)\end{pmatrix}$$

$$=\begin{pmatrix}\frac{1}{2^n}(M_{n+1}+2M_ni) & -\frac{1}{2^n}(M_n+2M_{n-1}i)\\ \frac{1}{2^{n-1}}(M_{n+1}+2M_ni) & -\frac{1}{2^{n-1}}(M_{n-1}+2M_{n-2}i)\end{pmatrix}$$

$$=\begin{pmatrix}-2.\left(-\frac{1}{2^{n+1}}M_{n+1}\right) & -2.\left(-\frac{1}{2^n}M_n\right)\\ -2.\left(-\frac{1}{2^n}M_n\right) & -2.\left(-\frac{1}{2^{n-1}}M_{n-1}\right)\end{pmatrix}+\begin{pmatrix}-2.\left(-\frac{1}{2^n}M_n\right) & -2.\left(-\frac{1}{2^{n-1}}M_{n-1}\right)\\ -2.\left(-\frac{1}{2^{n-1}}M_{n-1}\right) & -2.\left(-\frac{1}{2^{n-2}}M_{n-2}\right)\end{pmatrix}.i$$

$$=\begin{pmatrix}-2M_{-(n+1)} & M_{-n}\\ -2M_{-n} & M_{-(n-1)}\end{pmatrix}+\begin{pmatrix}-2M_{-n} & M_{-(n-1)}\\ -2M_{-(n-1)} & M_{-(n-2)}\end{pmatrix}.i$$

$$=IM_n^{-1}+IM_{n-1}^{-1}i=\begin{pmatrix}0 & 1\\ -2 & 3\end{pmatrix}^{-n}+\begin{pmatrix}0 & 1\\ -2 & 3\end{pmatrix}^{-(n-1)}i$$

$$=\begin{pmatrix}0 & 1\\ -2 & 3\end{pmatrix}^{-n}+\begin{pmatrix}0 & 1\\ -2 & 3\end{pmatrix}^{-n}\begin{pmatrix}0 & 1\\ -2 & 3\end{pmatrix}^{1}i$$
$$=\begin{pmatrix}0 & 1\\ -2 & 3\end{pmatrix}^{-n}\left[\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}+\begin{pmatrix}0 & 1\\ -2 & 3\end{pmatrix}i\right]$$
$$=\begin{pmatrix}0 & 1\\ -2 & 3\end{pmatrix}^{-n}\begin{pmatrix}1 & i\\ -2i & 1+3i\end{pmatrix}$$

Thus, we determine the following matrix decomposition

$$GIM_{-n}=\begin{pmatrix}0 & 1\\ -2 & 3\end{pmatrix}^{-n}\begin{pmatrix}1 & i\\ -2i & 1+3i\end{pmatrix},$$

for every integer $n\geq 0$.

Again, with the computational resource, in figures 3 and 4 we show other examples of data that confirm an invariant and mathematical behavior of interest in our investigation of the Gaussian numbers of Mersenne.

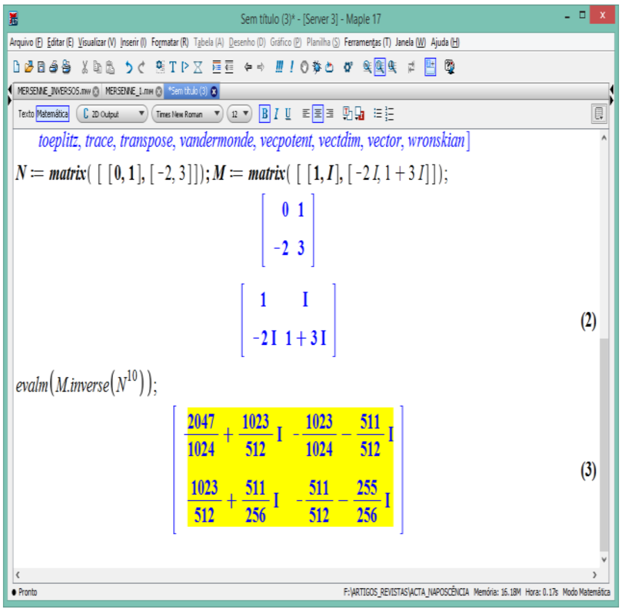

Figure 3. With the help of the CAS Maple we can determine high potencies of matrices related to the Gaussian Mersenne sequence (Prepared by the authors)

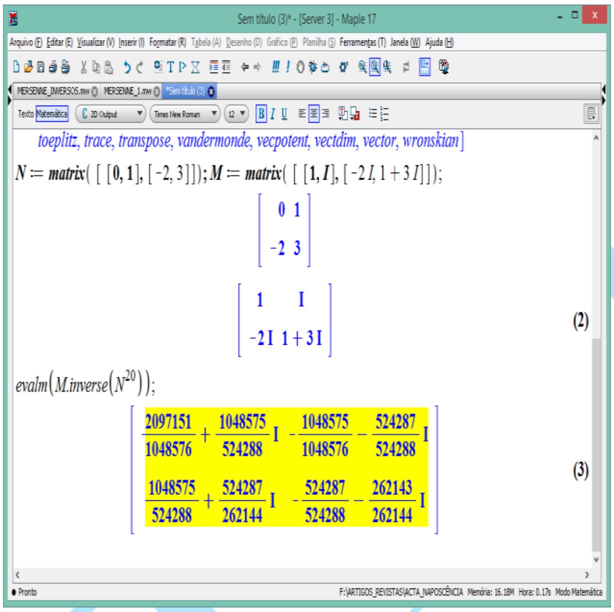

Figure 4. With the help of the CAS Maple we can determine high potencies of matrices related to the Gaussian Mersenne sequence (Prepared by the authors)

4. SOME IMPLICATIONS FOR THE TEACHERS IN BRAZIL

In the past sections, we have presented and detailed several matrix properties that can be explored in detail with the help of CAS Maple. Of particular form, we verified the possibility of discovering properties that can be enunciated, from the point of view of Mathematics, as true theorems. We observed that the activity of producing conjectures in the investigative process is essential.

From a historical and epistemological point of view, Mathematics teachers need to understand that Mersenne's Gaussian numbers, from the introduction of an imaginary unit, constitute a natural generalization of the original numbers of the Mersenne sequence. Thus, the notion of Gaussian numbers of Mersenne can acquire a practical operational and mathematical sense, inasmuch as in addition to being exposed to the proposition of a new mathematical definition, mathematics teachers in Brazil can operate and perform arithmetic and algebraic manipulations with the resource computational.

The mathematics teachers should understand that the combinatorial properties that can be determined with CAS Maple, a broad set of particular cases for the behavior of the matrix that we indicate by $IM_n = \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix}^n$ and $IM_{-n} = \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix}^{-n}$, where $n \geq 0$, allow the verification of the behavior of numbers of the Mersenne sequence for extremely large indices.

Similarly, mathematics teachers should understand that the combinatorial properties that can be determined with CAS Maple, a broad set of particular cases for the behavior of the matrix that we indicate by $\begin{pmatrix} 1 & i \\ -2i & 1+3i \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix}^n$ and $\begin{pmatrix} 1 & i \\ -2i & 1+3i \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix}^{-n}$, where $n \geq 0$, allow the verification of the behavior of numbers of the Gaussian Mersenne sequence for extremely large indices.

We observed the importance of an understanding by the Mathematics teacher regarding the evolutionary process of second-order recurrent sequences and that, in Brazil, almost predominantly; teachers maintain contact only with the Fibonacci sequence. On the other hand, from the study of the Mersenne sequence and to a certain extent the representation in the complex plane corresponding to the Gaussian sequence of Mersenne, teachers acquire a differentiated mathematical culture ([Kle12]; [Sti89]).

We find many books of History of Mathematics that usually emphasize a ludic, biographical and episodic aspect about the Fibonacci sequence, for example. When they recall, for example, anecdotal and picturesque aspects, such as the history of reproduction of immortal rabbit pairs, as recalled in the work ([AA18]) and ([Gul97]). We can understand his point of view from the figure below.

On the other hand, we seek to develop in Brazil investigations in the field of History of Mathematics with a strong interface with technology, providing teachers of Mathematics in initial formation a point of view and understanding of both the past, the moment of genesis of mathematical scientific concepts, and as, the current evolutionary stage of mathematical models and their increasing progress ([Alv18]).

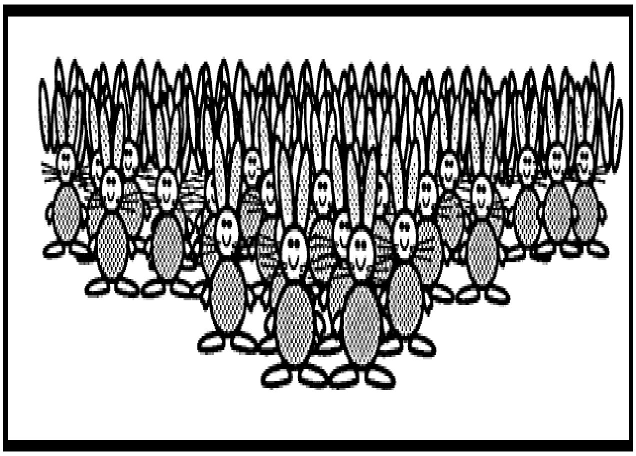

Figure 5. Gullberg ([Gul97], p.286) describes the sequence of immortal rabbits

5. CONCLUSIONS AND FUTURE RESEARCH

In this work, we discuss and detailed some properties derived from the second - order recursive sequence, which characterizes the Mersenne sequence defined by the recurrence relation $M_{n+2} = 3M_{n+1} - 2M_n$. It is worth noting that although the publication of results and more specialized properties only in scientific journals, the dissemination and knowledge of historical and epistemological value regarding an evolutionary character of the Mersenne´s model is very important because we can understand that Mathematics is made up of non-static scientific knowledge.

In the present work we introduce two new mathematical notions in the literature. The first involves the matrix representations that determine the numbers of the Mersenne sequence, for both positive integer and negative integer indices.

In addition, in an unprecedented way, we describe the elements of the sequence of Gaussian of Mersenne not yet formulated in the scientific literature and that, with the computational resource; we can identify several other interesting algebraic combinatorial mathematical properties.

In the table below we can see (on the left side) the formulation of several properties and the description of theorems that have been discovered from the use of CAS Maple, by verifying and testing a large set of data and invariant mathematical elements.

Table 1. Summarized information determined and related to the discovery of new theorems related to the notion of Mersenne number and Gaussian number of Mersenne

Description of discovered properties and theorems with the Maple´s help	Mathematical meaning
$IM_n = \begin{pmatrix} -2M_{n-1} & M_n \\ -2M_n & M_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix}^n, n \geq 0.$	It allows to determine the matrix constituted of the Mersenne numbers M_n from the powers of the generating matrix $\begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix}$.
$IM_{-n} = \begin{pmatrix} -2M_{-n-1} & M_{-n} \\ -2M_{-n} & M_{-n+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix}^{-n}, n \geq 0.$	It allows determining the matrix constituted of the Mersenne numbers from the powers of the generating matrix $\begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix}$, with the determination of negative sequence index elements M_{-n}.
$IM_{n+m} = IM_n . IM_m = IM_m . IM_n$ and $IM_{-(n+m)} = IM_{-n} . IM_{-m} = IM_{-m} . IM_{-n}$, for every integer $n \geq 0$.	Description of commutative properties with both positive and negative integer indices IM_{-n}, IM_{-m}.
$GIM_n = \begin{pmatrix} -2GM_{n-1} & GM_n \\ -2GM_n & GM_{n+1} \end{pmatrix} = \begin{pmatrix} 1 & i \\ -2i & 1+3i \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix}^n, n \geq 0.$	Description of matrices generating Gaussian numbers of Mersenne, with positive integer indices.
$GIM_{-n} = \begin{pmatrix} -2GM_{-n-1} & GM_{-n} \\ -2GM_{-n} & GM_{-n+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix}^{-n} \begin{pmatrix} 1 & i \\ -2i & 1+3i \end{pmatrix}, n \geq 0.$	Description of matrices generating Gaussian numbers of Mersenne, with negative integer indices.

Finally, we present in the present paper some elements that provide the mathematics teacher in Brazil not only the understanding of a process of generalization of the Mersenne sequence, as well as an important role of technology use as a significant interface to the History of Mathematics ([Sti89]) that cannot be to restrict to a simple set of episodic stories transmitted to the teacher of Mathematics in initial formation in the academic context and that do not become something actually embedded in his classroom practice and teaching of Mathematics.

Regarding our future research and other implications, we can define the following matrix, the entries of which are to be described by means of the Mersenne quaternions $QIM_n = \begin{pmatrix} -2QM_{n-1} & QM_n \\ -2QM_n & QM_{n+1} \end{pmatrix}$. A quaternion of Mersenne will be defined by the following relation: $QM_n = M_n + M_{n+1}i + M_{n+2}j + M_{n+3}k$, where the imaginary units have the propertie $i^2 = j^2 = k^2 = -1$.

Repeating some of the previous arguments, we can verify that:

$$QIM_n = \begin{pmatrix} -2QM_{n-1} & QM_n \\ -2QM_n & QM_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix}^n . M_Q,$$

where

$$M_Q = \begin{pmatrix} -2\left(-\frac{1}{2} + 0i + 1j + 3k\right) & 0 + i + 3j + 7k \\ -2(0 + 1i + 3j + 7k) & 1 + 3i + 7j + 15k \end{pmatrix}.$$

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