ALTERNATIVE VIEWS OF SOME EXTENSIONS OF THE PADOVAN SEQUENCE WITH THE GOOGLE COLAB

Renata Passos Machado Vieira [1], Francisco Regis Vieira Alves [2], Paula Maria Machado Cruz Catarino [1]

[1] **Federal Institute of Education, Science, Technology of Ceará - Brazil, Department of Mathematics**
[2] **University of Trás-os-Montes - Portugal, Department of Mathematics**

Corresponding Author: Renata Passos Machado Vieira, re.passosm@gmail.com

ABSTRACT: The article discusses an alternative way of viewing Padovan sequence extensions through Newton fractals using Google Colab to develop these reproductions. Initially, a scientific bibliographic survey is conducted regarding the process of extension of this sequence. Thus, studies are made on this qualitative alternative of visualization for linear and recurrent sequences. Hereafter alternative forms are developed in Google Colab, so each fractal generated from the characteristic polynomial of these numbers is analyzed. Finally, the conclusions about this process are discussed, proposing studies to continue this work in the future and research in Brazil, as well as possible applications in the areas of Physics and the formation of teachers of Natural Sciences in Brazil.
KEYWORDS: Extension, Fractal, Google Colab, Padovan sequence, Visualization.

1. INTRODUCTION

The Padovan sequence is a homogeneous, linear and recurring sequence, of third order, originally created by the Italian architect Richard Padovan. Born in 1935, this architect studied architecture at the Architectural Association in London (1952-57). Since then, he has combined practice with teaching and writing in architecture. He believes, however, that his true architecture education began when he found the work and thought of Dutch architect Dom Hans van der Laan in 1974 ([Ste96]). This dutch member of the Benedictine order studied the plastic number, which is the only real solution to the characteristic cubic equation of the Padovan sequence, where its roots are known as the family of plastic numbers ([SB09]).
Thus, it is necessary to know the recurrence formula and its characteristic polynomial, described below, through the following mathematical definition.
Definition 1. The recurring sequence of Padovan, with initial terms given by $P_0 = P_1 = P_2 = 1$ e $n \in \mathbb{N}$, is defined by the relation:

$$P_n = P_{n-2} + P_{n-3}, n \geq 3. \quad (1)$$

From this recurrence relationship, we can verify the following properties that we indicate below:

$$\frac{P_n}{P_{n-2}} = \frac{P_{n-2}}{P_{n-2}} + \frac{P_{n-3}}{P_{n-2}}$$
$$\frac{P_{n-1}}{P_{n-1}} \cdot \frac{P_n}{P_{n-2}} = 1 + \frac{P_{n-3}}{P_{n-2}}$$
$$\frac{P_{n-1}}{P_{n-1}} \cdot \frac{P_n}{P_{n-2}} = 1 + \frac{1}{\frac{P_{n-2}}{P_{n-3}}}$$

According to studies and research conducted regarding the convergence relationship between the terms of the Padovan sequence, the following limit can be determined. ([Alm14] and [PF08]):

$$\lim_{n\to\infty} \frac{P_{n+1}}{P_n} \approx 1.32 = \psi. \quad (2)$$

Thus, one can obtain the characteristic equation of the Padovan sequence that follows:

$$\frac{P_{n-1}}{P_{n-1}} \cdot \frac{P_n}{P_{n-2}} = 1 + \frac{1}{\frac{P_{n-2}}{P_{n-3}}}$$
$$\psi^2 = 1 + \frac{1}{\psi}$$
$$\psi^3 - \psi - 1 = 0$$

Now, let us note that this polynomial equation $\psi^3 - \psi - 1 = 0$ has exactly three roots ψ_1, ψ_2, ψ_3, two of which are complex and conjugate roots and only one real root given by the approximate number of 1.32, known in the scientific literature as the plastic number.
This sequence also has some forms of algebraic representations, which are then found in the works of Vieira and Alves ([VA18]) and Vieira and Alves ([VA19a]).
This recurring sequence also has some other forms of algebraic representations, which are then found in the works of Vieira and Alves ([VA18]) and Vieira

and Alves ([VA19a]).

One can note a 3D geometric sequence representation created on Wolfram Software (Figure 1). This representation supports the sequence properties and theorems investigation in initial mathematictian formation teacher's course. It's worth to emphatize a good background on padovan numbers and its epistemical field is required, always assuring the mathematic rigor on geometric contruction and conjecturs definitions.

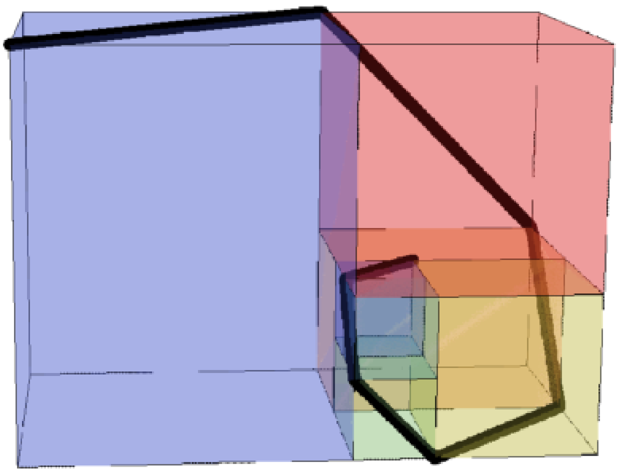

Figure 1. Geometrical representation of the Padovan spiral in 3D. Prepared by the authors

Based on this, the other sections will discuss the extensions of this sequence from order 3, known as Padovan numbers, order 4, called the Tetradovan sequence, and up to order 5, called the Pentadovan sequence. However, a brief discussion is made about Newton fractals as an alternative to geometric visualization of recurring linear sequences, as we will discuss here the case of the Padovan Sequence and some of its generalizations that have been recently introduced in the literature.

Finally, the Padovan sequence fractals, according to their corresponding characteristic polynomial, are generated through Google Colab and the results obtained with the aid of this computational tool are analyzed and discussed, allowing a greater mathematical understanding of the subject.

2. SOME EXTENSIONS OF THE PADOVAN SEQUENCE

As with the emblematic Fibonacci sequence ([Cer15], [Mil71]), the Padovan sequence can be extended according to its order of recurrence, starting from the third order, known as the Padovan numbers.

Thus, for an order 4, we have those called by Tridovan numbers. Below we present the corresponding mathematical definition.

Definition 2. The Tridovan sequence has the following recurrence formula for $n \in \mathbb{N}$ and the initial values indicated by $Tr_0 = 1, Tr_1 = 0, Tr_2 = Tr_3 = 1$ ([VA19a]):

$$Tr_n = Tr_{n-2} + Tr_{n-3} + Tr_{n-4}, n \geq 4. \qquad (3)$$

Similarly, from definition 2, it is possible to perform algebraic manipulations to obtain the characteristic polynomial of this sequence, as indicated below.

$$\frac{Tr_n}{Tr_{n-3}} = \frac{Tr_{n-2}}{Tr_{n-3}} + \frac{Tr_{n-3}}{Tr_{n-3}} + \frac{Tr_{n-4}}{Tr_{n-3}}$$

$$\frac{Tr_{n-1}}{Tr_{n-1}} \cdot \frac{Tr_{n-2}}{Tr_{n-2}} \cdot \frac{Tr_n}{Tr_{n-3}} = \frac{Tr_{n-2}}{Tr_{n-3}} + 1 + \frac{1}{\frac{Tr_{n-3}}{Tr_{n-4}}}$$

Now analyzing the convergence relationship of this extension indicated by Tr_n. It can be seen, as shown in Table 1, that by dividing a term by its previous one, this new sequence should converge to the real value indicated by 1.46.

Table 1. Convergence analysis between Tridovan's neighboring terms. Prepared by the authors

n	Tr_n	$\frac{Tr_{n+1}}{Tr_n}$
0	1	0
1	0	∄
2	1	1
3	1	2
4	2	1
5	2	2
6	4	1.25
7	5	1.6
8	8	1.37
9	11	1.54
10	17	1.411
11	24	1.5
12	36	1.44
13	52	1.48
14	77	1.45
15	112	1.46
16	165	1.46

This way we can determine that:

$$\lim_{n \to \infty} \frac{Tr_{n+1}}{Tr_n} \approx 1.46 = \mu \in \mathbb{R}. \qquad (4)$$

Thus, we can return to and repeat the same process of algebraic manipulation, obtaining its respective characteristic polynomial and which we indicate below:

$$\frac{Tr_{n-1}}{Tr_{n-1}} \cdot \frac{Tr_{n-2}}{Tr_{n-2}} \cdot \frac{Tr_n}{Tr_{n-3}} = \frac{Tr_{n-2}}{Tr_{n-3}} + 1 + \frac{1}{\frac{Tr_{n-3}}{Tr_{n-4}}}$$

$$\mu^3 = \mu + 1 + \frac{1}{\mu}$$
$$\mu^4 - \mu^2 - \mu - 1 = 0$$

However, this equation has exactly four roots that we will denote by $\mu_1, \mu_2, \mu_3, \mu_4$ (two complex ones, one positive real and one negative real root).

We emphasize that the positive real root has a value of approximately 1.46, which is the same value of the convergence relationship between the neighboring terms of this sequence.

Consequently, for a homogeneous recurring sequence of order 5, called Tetradovan, it´s fundamental recurrence is then defined ([VA19a]). Below we present its mathematical definition.

Definition 3. For every positive integer $n \in \mathbb{N}$, with the initial conditions indicated by $Te_0 = 1, Te_1 = 0, Te_2 = Te_3 = 1, Te_4 = 2$, the recurrence of the Tetradovan sequence is given by [VA19a]):

$$Te_n = Te_{n-2} + Te_{n-3} + Te_{n-4} + Te_{n-5}, n \geq 5. \quad (4)$$

By repeating some previous algebraic arguments that we will now apply, according to definition 3, we can determine that:

$$\frac{Te_n}{Te_{n-4}} = \frac{Te_{n-2}}{Te_{n-4}} + \frac{Te_{n-3}}{Te_{n-4}} + \frac{Te_{n-4}}{Te_{n-4}} + \frac{Te_{n-5}}{Te_{n-4}}$$
$$\frac{Te_{n-1}}{Te_{n-1}} \cdot \frac{Te_{n-2}}{Te_{n-2}} \cdot \frac{Te_{n-3}}{Te_{n-3}} \cdot \frac{Te_n}{Te_{n-4}}$$
$$= \frac{Te_{n-3}}{Te_{n-3}} \cdot \frac{Te_{n-2}}{Te_{n-4}} + \frac{Te_{n-3}}{Te_{n-4}} + 1 + \frac{1}{\frac{Te_{n-5}}{Te_{n-4}}}$$

Analyzing in a similar way as done for Tridovan numbers, we can establish the convergence relationship between the terms immediately next to the sequence, according to table 2 below.

Therefore, it can be established that the following value for the limit involving the quotient of the terms: $\frac{Te_{n+1}}{Te_n}$:

$$\lim_{n\to\infty} \frac{Te_{n+1}}{Te_n} \approx 1.53 = \varpi \in \mathbb{R}. \quad (5)$$

Thus we can obtain the corresponding or associated characteristic polynomial equation of this sequence as follows:

$$\varpi^4 = \varpi^2 + \varpi + 1 + \frac{1}{\varpi}$$
$$\varpi^5 - \varpi^3 - \varpi^2 - \varpi - 1 = 0$$

Table 2. Convergence analysis between Tetradovan's neighboring terms. Prepared by the authors

n	Te_n	$\frac{Te_{n+1}}{Te_n}$
0	1	0
1	0	∄
2	1	1
3	1	2
4	2	1.5
5	3	1.33
6	4	1.75
7	7	1.42
8	10	1.6
9	16	1.5
10	24	1.5416
11	37	1.5405
12	57	1.5263
13	87	1.54
14	134	1.529
15	205	1.53
16	315	1.53

Let us now note that this equation has five roots denoted by $\varpi_1, \varpi_2, \varpi_3, \varpi_4, \varpi_5$, being four complex roots and only one real root, where this real root has a positive value of approximately 1.53, the same value as the ratio of $\frac{Te_{n+1}}{Te_n}$.

Some properties inherent in these numbers originated in the Tridovan sequence may be studied in more detail from the recent work of Vieira and Alves ([VA19b]).

In the next and main section of the paper, Newton's fractals will be studied as a way of geometrically visualizing the linear and recurrent sequences from their corresponding characteristic polynomial, transforming them into functions.

3. NEWTON´S FRACTAL AS A WAY OF VISUALIZING PADOVAN'S LINEAR AND RECURRENT SEQUENCES

There are many ways to get the roots of a mathematical equation, depending on its degree, where for 2nd degree equations we know the Bhaskara method. For 3rd degree equations, there is the classic method of Girolamo Cardano (1501 - 1576), but for degrees higher than the fourth order, Niels Henrik Abel (1802 - 1829) proved that it is not possible to solve equations by radicals and combinations of coefficients. ([CM00]).

Thus, numerical mathematical methods and computational resources are used to obtain the roots of these equations of degree greater than four or approximately.

Newton's method is well known for calculating successive numerical approximations of polynomial function zeros and can converge quickly or slowly if the iteration begins near the desired root. But when the iteration begins far from the desired root, care must be taken not to make representative errors in this method ([TA15]).

An example of this method is shown, for example, in the work of Theodore and Aguilar ([TA15]). Thus, by performing this technique for the Padovan sequence, we can transform the corresponding characteristic polynomial into the function $f(\psi) = \psi^3 - \psi - 1$ and then graph that function as shown in Figure 2. It is thus observed that there is only one real root in the interval [1,2].

The use of the computational instrument allows the exploration of a broad set of mathematical qualitative properties from figure 2.

Applying Newton's method to the indicated function $f(\psi) = \psi^3 - \psi - 1$, we have to:

$$N(\psi) = \psi - \frac{f(\psi)}{f'(\psi)} = \psi - \frac{\psi^3 - \psi - 1}{3\psi^2 - 1} = \frac{2\psi^3 + 1}{3\psi^2 - 1} \quad (6)$$

It then decides an initial choosen value $\psi_0 = 1$ (approximation) for the actual root from the defined a priori interval, then:

$$\psi_0 = 1$$
$$\psi_1 = 1.5$$
$$\psi_2 = 1.34$$
$$\psi_3 = 1.3252$$
$$\psi_4 = 1.3247181$$
$$\psi_5 = 1.324717957244789 \ldots$$
$$\psi_6 = 1.32471795724474602596091 \ldots$$
$$\psi_7 = 1.32471795724474602596091 \ldots$$

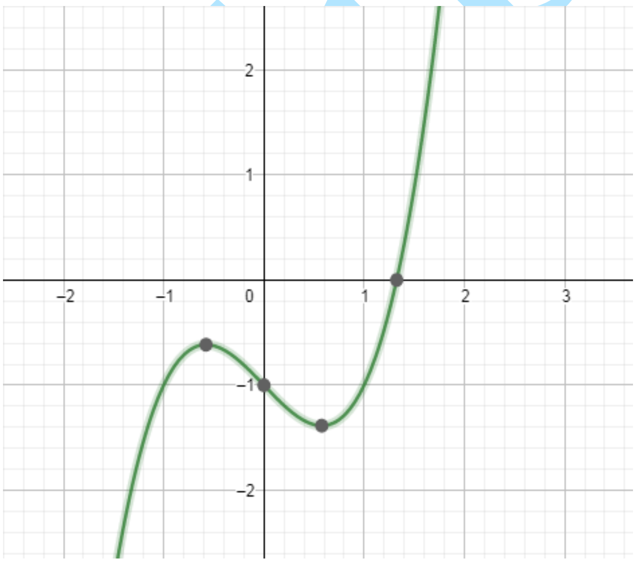

Figure 2. Graph view of the polynomial function. Prepared by the authors

From the sixth interaction, one notices a precision in the result, obtaining the real root of the function. For the other cases of Padovan sequence extensions described in the previous sections, the same procedure is performed to obtain the roots using the Newton method.

Fractals are of great importance in the area of mathematical research and can assume a multitude of shapes and difficult solutions ([Bur09]). According to Carreira and Andrade ([CA10]), fractals are characterized by having at least one of the following properties: a) self-resemblance - where a piece of the fractal is similar to the rest of the figure; b) infinite complexity - presenting an infinite number of interactions in the fractal; c) fractal dimension - unlike Euclidean geometry, the fractal does not always have the entire dimension and can be fractional and/or infinite dimension.

These fractals can be further classified according to how they are generated ([Alv07]): a) fractals defined by a recurrence relationship at each point in space, checking whether or not the sequence tends to infinity; b) random fractals that are generated by stochastic processes; c) fixed point of an iterated function system, where the sets of functions are applied successively.

The application of the fractals is performed according to the modeling of the theoretical objects of interest and comparing with their fractal geometry, considering several approximation factors, depending on their degree of accuracy.

These fractals can be applied in several areas of science, highlighting the area of mathematics, and it is possible to analyze even some linear and recurring sequences. A linear and recurring sequence is one in which there is an infinite number of terms, which are generated by a linear recurrence, called the recurrence formula ([Zie59]), obtaining a load that allows the calculation of its immediate predecessor terms.

However, this recurrence is not the only way to define linear and recurrent sequences, and it is still necessary to know their initial elements.

This type of sequence has several orders with constant coefficients that are recurrent from the formula: $x_{n+3} + rx_{n+2} + px_{n+1} + qx_n = 0$, where $r, p, q \in \mathbb{Z}$ and $q \neq 0$ or $q \neq 0$, for the case $q = 0$ or $p = 0$.

Therefore, the sequence will become first or second order ([Lim76]). Thus, every sequence has a characteristic polynomial that varies according to its order, making it necessary to calculate its roots.

Thus, one of the ways to know and determine the root of a mathematical equation of a linear and recurring sequence can be through Newton's fractal method, where it is possible to visualize the roots with this computational resource, and also diagnose applications in various areas of science and mathematics such as nature science and mathematics

An example is the case of the Fibonacci sequence, where in Monnerot-Dumaine's work ([Mon09]) its fractal is constructed and its results analyzed, thus creating various properties and patterns according to the formed figure.
Using the Newton fractal we introduced in this work, involving an alternative visualization of linear and recurring sequences, one can analyze and understand the behavior of their roots and qualitatively diagnose their application in everyday life, since many formulas and mathematical definitions of these sequences are sometimes conjectured, without knowing their real application and visualization or geometric interpretation.
Newton's fractal is, therefore, a fractal art in which it transforms and modifies the algebraic representations of abstract mathematical functions into a medium that can be dynamic images, music, and animations.
In the following section, this method is performed with the Google Colab tool, from application to Padovan sequence, the Tridovan sequence and the Tetradovan sequence, as well as some discussions of their generated fractal fractals.

4. VISUALIZATION WITH THE GOOGLE COLAB´S AID

Google Colab is a free tool hosted by Google that runs essentially in the cloud. This feature allows you to perform computational simulations with support for Python and some libraries already installed, saving files to Drive and the setup process quickly and easily.
For the present work, the choice of this tool for mathematical computing was based on the fact that this software is free, easy to use, and has cloud hosting. Thus, the images of abstract functions will be generated from this computational resource.
Figures 3, 4, 5 and 7 were plotted and presented below. They are characterized as fractals made through a recurrence at each point of their representation space.
Thus, the Phyton code was developed to plot the fractals according to Newton's method and, with the help of the characteristic equation of the corresponding sequence, then shown in the figures below, and analyzed according to their pockets and lobes.
Initially, for the function of the Padovan sequence that we determine by $(f(\psi) = \psi^3 - \psi - 1)$, we have the following generated Newton fractal, as shown in figure 3. Thus, there are three pockets, one more central and two more lateral, varying according to the amount of roots.
Note also the presence of three ellipses filled with other ellipses within this, called lobes. These lobes represent the corresponding roots of the function, where by drawing a horizontal line in the middle of the figure, only one horizontal lobe is found. Thus, we visualize the existence of only one real root, being approximately the real value indicated by 1.32.

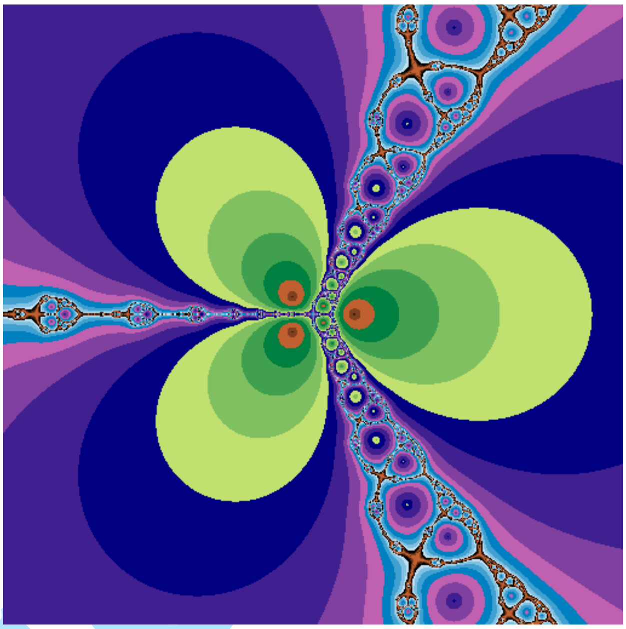

Figure 3. Newton's fractal of the Padovan sequence. Prepared by the authors

Next, building Newton's fractal from the Tridovan sequence, from its function found in the previous section and which we wrote $(f(\mu) = \mu^4 - \mu - \mu - 1)$, we have, as shown now in figure 4, the existence of four pockets, which are then generated from the branching of the central pocket found in the Padovan fractal (see figure 3) of the sequence.

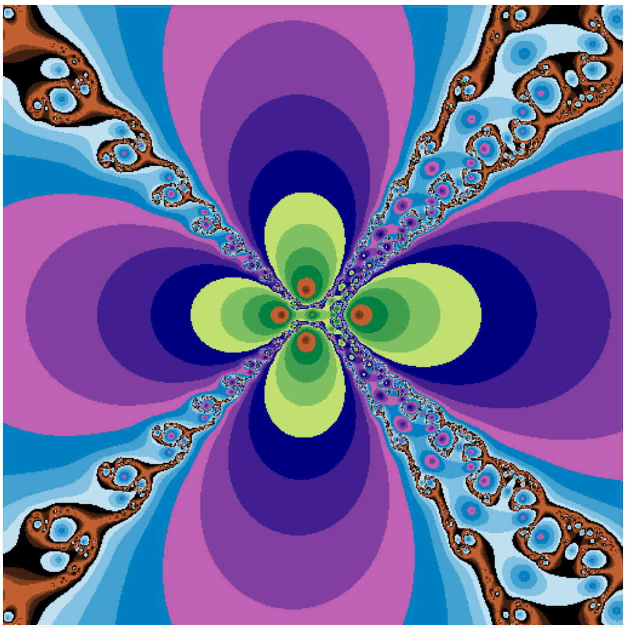

Figure 4. Newton's fractal of the Tridovan sequence. Prepared by the authors

We also visualized the presence of four lobes, thus drawing a horizontal and central line in the figure, and verifying the existence of two real roots. It is observed that one is on the left and right side, the one on the left side represents the negative real root and the one on the right side represents the positive real root, the latter being previously studied and having an approximate real value of 1.46. The others that are located on the vertical axis are the complex roots. We also notice a decrease in the orange lobe, as we can see next.

Finally, we have the construction of Newton's fractal Tetradovan sequence from its function ($f(\varpi) = \varpi^5 - \varpi^3 - \varpi^2 - \varpi - 1$), which is represented by Figure 4. In this case, we notice the presence of five pockets, one more central and the other four more lateral, representing the number of roots of the characteristic equation. This fractal also has five lobes, representing the roots of the function. Analyzing them by drawing a horizontal line in the center of the figure 5, there is only one real root and the others that are outside that axis are the complex roots. This real root, as seen in the previous section, has its approximate value of 1.53 and can be found through the qualitative analysis of the figure. Note also the decrease of the orange lobe.

By inserting the fractal figures into the Cartesian plane of representation, we can estimate and verify the root value, verifying that the furthest colors from orange will be the furthest values from the root values. Already the values closest to the orange lobe will be the closest to zero, that is, closest to the values of the roots of the function that we are analyzing.

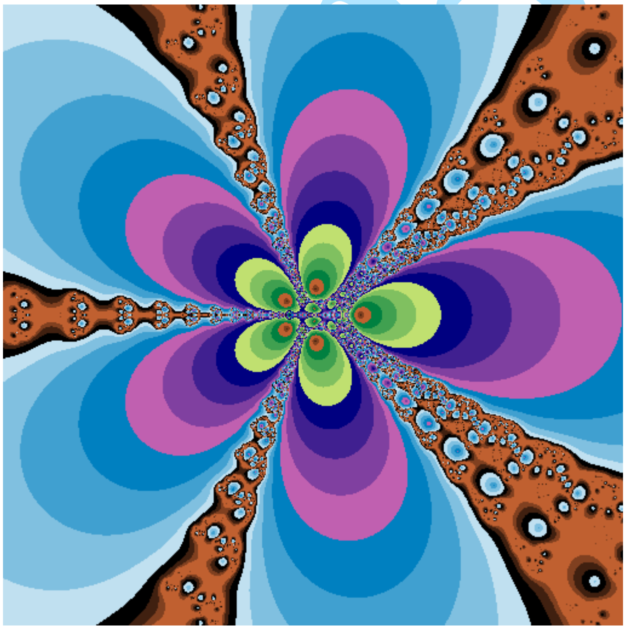

Figure 5. Newton's fractal of the Tetradovan sequence. Prepared by the authors

This process of root-grid search, lobes, represents Newton's method. Performing successive substitutions to find the root of the function, this method using the Google Colab computational resource facilitates the calculation and obtaining of roots and, with the support of visualization, the understanding of expressly algebraic and formal properties.

Given this analysis developed in the current section, one can establish a certain pattern of growth of the pockets, where initially there are three pockets, one central and two laterals, in which from the extensions, this central pouch will divide, transforming it in two bags.

When the central pouch does not exist, it means that it has already been divided into the previous extension, appearing again in the next extension of the recurrent sequence we analyzed. We also highlight that the quantity of pockets represents the quantity of roots of the characteristic polynomial associated with the numerical sequence.

Therefore, the lobes represent the roots of the function, further verifying their real roots and complex roots by drawing a horizontal line in the center of the figure. Thus, the lobes located on the horizontal axis will represent the real roots, and those on the vertical axis will represent the complex roots.

Bringing Newton's fractals from the Padovan, Tridovan, and Tetradovan sequence to objects in nature, we can see similarities of these with flowers seen from a top-down perspective, where their lobes represent petals and sepals.

Thus, Newton's fractal of the Padovan sequence represents a threefold flower with three petals and three sepals; Newton's fractal of the Tridovan sequence represents a four-petal four-petal tetramer flower; and Newton's fractal of the Tetradovan sequence is a pentameric flower with five petals and five sepals ([SB03]).

It is also noted that the resemblance to the geranium flower, varying according to its petals. It can be seen in figure 6 this flower with four petals, with rounded corners, approaching an ellipse. It is also important to note the qualitative alternations of flower colors, as happens in the lobes of the figures discussed above.

This species belongs to a group of small shrub herbs of the genus *Geranium* and *Pelargonium*, also known as Sardines. Gathering about 300 species, these flowers are found in the temperate and tropical highlands, many of which are cultivated as ornamental or for tannin and tincture extraction.

To conclude, it is noteworthy that these figures generated by Google Colab present a form of visualization, and can be seen from another angle different from that shown in the paper and with important implications for the teaching and scientific dissemination of mathematical knowledge in Brazil.

Figure 6. Geranium flower Source: Taken from Pixabay website

5. CONCLUSIONS

Thus, as happens in the works of Popoola, Dawody and Yusuf ([PDY18]) and Alves, Catarino and Mangueira ([ACM19]), showing their application and using computational resources to facilitate the procedure, it becomes increasingly necessary to know the use and application of mathematical contents, which are often studied only in pure mathematics articles, not investigating their application.

The present work showed a way to visualize the roots of the characteristic equation of the sequence of Padovan, Tridovan, and Tetradovan, also making discussions and analyzes about their Newton fractals, generated from Google Colab. This method facilitates the visualization of the behavior of Newton's numerical algorithm to calculate the roots of the characteristic equation of the linear and recurrent sequence.

Given the investigative process carried out in this work, it was possible to notice similarity with the figures presented during this work, with the flowers existing in our daily lives, generating a color image from this abstract mathematical equation.

In future research, we seek to generalize the Padovan sequence based on its extension order, as well as its application in the areas of Physics and Natural Sciences.

For this, in addition to the discussions about Newton's fractals of these sequences, we can see in figure 7 Newton's fractal of the sixth order Pentadovan sequence, as shown in figure 6.

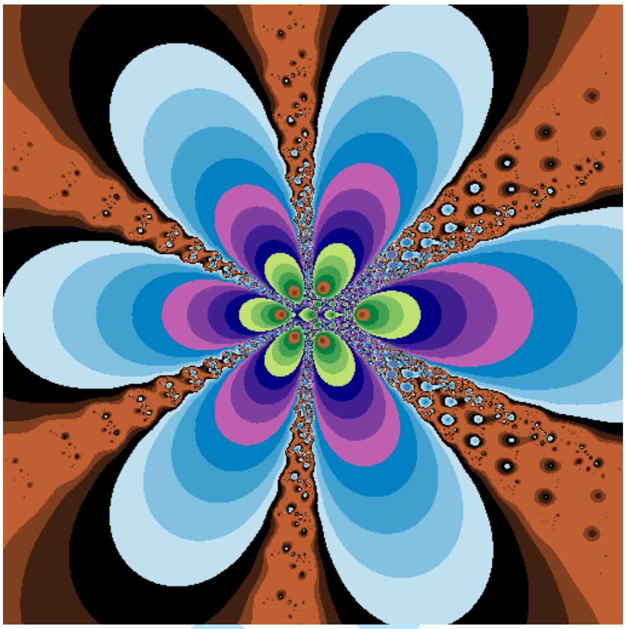

Figure 7. Newton Fractal of the Pentadovan Sequence. Prepared by the authors

Repeating the qualitative visualization process, based on this figure, we found the analysis of the number of pockets and lobes linked to the roots of the sequence function. Thus, for this sixth-order sequence, there will be six roots as shown in figure 6. Noting further that the orange lobe will become increasingly distant.

Concerning systematic study and research under development in Brazil, we present the mathematical properties of sequences recently introduced in the scientific literature, named Padovan sequence, Tridovan sequence, Tetradovan sequence, Pentadovan sequence, Hexadovan sequence, etc. and, finally, sequence, generalized z-Dovan. The mathematical results are recent, however, the advance of the research developed in Brazil should provide other forms of its study and analysis. In this case, Google Colab also has repercussions for teacher education in Brazil.

REFERENCES

[Alm14] **E. G. S. Almeida** – *Propriedades e generalizações dos números de Fibonacci*. Tese (Doutorado), 2014.

[Alv07] **C. M. F. S. J. Alves** – *Fractais: conceitos básicos, representações gráficas e aplicações ao ensino não universitário*. Tese de mestrado, departamento de matemática, universidade de Lisboa, 2007.

[ACM19] **F. R. V. Alves, P. M. M. C. Catarino, M. C. dos S. Mangueira** – *Discovering* *theorems about the gaussian mersenne sequence with the Maple´s help*. Annals. Computer Science Series, Vol. XVII (1), 2019.

[Bur09] **A. Burton** – *Newton's method and fractais.* Technical manuscript, Whitman College, 2009.

[Cer15] **J. L. Cereceda** – *Binet's formula for generalized tribonacci numbers.* International Journal of Mathematical Education in Science and Technology, v. 46, n. 8, p. 1235–1243, Apr. 2015.

[CA10] **A. S. Carreira, C. A. D. Andrade** – *Geometria a várias dimensões - Fractais*. Faculdade de Ciências, Universidade de Lisboa, 1999.

[CM00] **D. M. Cláudio, J. M. Marins** – *Cálculo Numérico Computacional – Teoria e Prática.* São Paulo: Atlas, 2000.

[Lim76] **E. L. Lima** – *Curso de análise*. Rio de Janeiro, Instituto de Matemática Pura e Aplicada, CNPq, v. 1, 1976.

[Mil71] **M. D. Miller** – *On generalized Fibonacci numbers.* The American Mathematical Monthly, v. 78, n. 10, p. 1108–1109, 1971.

[Mon09] **A. Monnerot-Dumaine** – *The Fibonacci Word fractal*. Hal - archives-ouvertes, 24 pages, 25 figures. 2009.

[PF08] **L. C. Pereira, M. V. Ferreira** – *Sequência de fibonacci: história, propriedades e relações com a razão áurea.* Disciplinarum Scientia Ciencias Naturais e Tecnológicas, v. 9, n. 1, p. 67–81, 2008.

[PDY18] **O. P. Popoola, O. O. Dawody, O. O. Yusuf** – *Quadratic regression and factorial analysis on the effect of climatic elements on global food production and land nutrients in Africa.* Annals. Computer Science Series, Vol. XVI (1), 2018.

[Ste96] **I. Stewart** – *Tales of a neglected number*. Mathematical Recreations – Scientific American, v. 274, p. 102–103, 1996.

[SB03] **J. H. R. Santos, H. Bonilla** – *Organografia das fanerógamas.* Fortaleza: UECE, 146 p., 2003.

[SB09] **V. M. W. de Spinadel, A. R. Buitrago** – *Towards van der laan's plastic number in the plane*. Journal for Geometry and Graphics, v. 13, n. 2, p. 163–175, 2009.

[S+94] **V. Sassone, M. Nielsen, G. Winskel, J. Doe** – *Relationships between models of concurrency*, Proceedings of IEEE, 1994.

[TA15] **M. M. Teodoro, J. C. Z. Aguilar** – *O método de Newton e fractais.* Dissertação de Mestrado Profissional em Matemática – Universidade Federal de São João del-Rei, 2015.

[VA18] **R. P. M. Vieira, F. R. V. Alves** – *Sequência de Padovan Afim e as suas propriedades.* Revista Thema, v. 15, n. 4, p. 1269-1276, 2018.

[VA19a] **R. P. M. Vieira, F. R. V. Alves** – *Propriedades das extensões da Sequência de Padovan*. Revista Eletrônica Paulista de Matemática, v. 15, p. 24-40, 2019, Edição de Iniciação Científica.

[VA19b] **R. P. M. Vieira, F. R. V. Alves** – *Sequences of Tridovan and their identities.* Notes on Number Theory and Discrete Mathematics, v. 25, n. 3, p. 185-197, 2019.

[Zie59] **N. Zierler** – *Linear recurring sequences*. Journal of the Society for Industrial and Applied Mathematics, v. 7, n. 1, p. 31–48, march 1959.