TOPP LEONE EXPONENTIAL – EXPONENTIAL DISTRIBUTIONS PROPERTIES AND APPLICATION

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ABSTRACT: In this paper, a new distribution called ToppLeone Exponential – Exponential distribution (TLEE) is developed by making Topp Leone Exponential G family as Generator while exponential distribution as the base line distribution. The respective density and distribution functions of this distribution were shown. Some structural properties of this new distribution were derived such as moment generating function, renyi entropy and order statistics.Maximum Likelihood Estimate (MLE) methods was used to estimate the parameters of this new distribution. Finally, two real data sets were used to validate the results obtained from MLE. The results shown that TLEE distribution provide better fit in the data sets than some other commonly known distributions. Perhaps, this new distribution may be used to modeling positive real life data sets.

KEYWORDS: Topp Leone Exponential – Exponential distribution; Moment generating function; Renyi entropy; Order statistics; maximum likelihood estimation.

1. INTRODUCTION

Statisticians had developed some generalizations of the exponential distribution that can provide some flexibility as exponential distribution is well known with constant failure rate and memory less property and limited to model some unimodal data. Some of the generalizations of exponential distribution developed are exponentiated-exponential (EE) distribution by [1], generalization of the exponential distribution introduced by [2], beta-exponential (BE) distribution developed by [3], exponentiated exponential distribution by [4], transmuted exponentiated exponential distribution by [5], gamma-exponentiated exponential distribution by [6], exponentiated exponential distribution by [7], beta generalized exponential distribution by [8] and exponentiated kumaraswamy exponential distribution by [9].

Furthermore, inverse exponential and some generalization of the inverse distribution have been developed which aim at improving on the flexibility of exponential distribution, such as inverted exponential (IE) distribution by [10]; therefore, the inverted exponential (IE) distribution has been generalized and extended to yield the generalized inverse exponential (GIE) distribution by [11], also exponential Inverse exponential (EIE) was developed by [12].

However, in this paper; we developed a new distribution called Topp Leone Exponential-Exponential (TLEE) distribution in order to improve on the flexibility of exponential distribution. The densities of TLEE are defined. Some structural properties of the new model are discussed, and the mathematical expressions are derived. Estimation of the model parameters by the method of maximum likelihood estimate (MLE) is discussed. The flexibility of this distribution is illustrated in an application to two real life data sets. The remaining sections of the paper; is organized as follows. The cdf, pdf and densities graphs of the new distribution are defined in section 2. Useful expansion of TLEE is discussed in section 3. Some mathematical properties of the TLEE are discussed in section 4. The maximum likelihood estimates for the parameters of TLEE are presented in section 5. Applications to two real data sets for the model are shown in Section 6 and Section 8 concludes the paper.

2. TOPP LEONE EXPONENTIAL – EXPONENTIAL DISTRIBUTION

In this section, we defined the cumulative distribution function (cdf) and probability density function (pdf) of the new distribution called Topp Leone Exponential – Exponential.

Thus, we developed and defined the cdf and pdf of Topp Leone Exponential G family of distributions as shown below; therefore, the cdf and pdf of the Topp Leone Exponential – G family of distributions (TLE-G) are defined as follows:

$$F_{TLE-G}(x;\sigma,\lambda,\beta) = \left[1 - exp\left\{-2\lambda\left(\frac{G(x;\beta)}{\bar{G}(x;\beta)}\right)\right\}\right]^{\sigma} \quad (1)$$

and

$$f_{TLE-G}(x;\beta) = \frac{2\sigma\lambda g(x,\beta)}{\left(\bar{G}(x;\beta)\right)^2} exp\left\{-2\lambda\left(\frac{G(x;\beta)}{\bar{G}(x;\beta)}\right)\right\}\left[1 - exp\left\{-2\lambda\left(\frac{G(x;\beta)}{\bar{G}(x;\beta)}\right)\right\}\right]^{\sigma-1} \quad (2)$$

Respectively but, the cdf and pdf of the Exponential distribution are given as follows:

$$G(x) = 1 - \exp\{-\gamma x\} \quad (3)$$

and

$$g(x) = \gamma \exp\{-\gamma x\} \quad (4)$$

$\vartheta, x \succ 0$, respectively.

Note: equation (5) below is considered, after the substitution of both equations (3) and (4) into (1) and (2).

$$\frac{1 - exp\{-\gamma x\}}{exp\{-\gamma x\}} = exp\{\gamma x\} - 1 \quad (5)$$

Therefore, the cdf and pdf of the new distribution called TLEE are respectively defined as

$$F_{TLE-E}(x;\sigma,\lambda,\gamma) = [1 - exp\{-2\lambda(exp\{\gamma x\} - 1)\}]^{\sigma} \quad (6)$$

$$f_{TLE-E}(x;\sigma,\lambda,\gamma) = \frac{2\sigma\lambda\gamma}{exp\{-\gamma x\}} exp\{-2\lambda(exp\{\gamma x\} - 1)\}\,[1 - exp\{-2\lambda(exp\{\gamma x\} - 1)\}]^{\sigma-1} \quad (7)$$

Henceforth, a random variable X with density function given in equation (7) follows $TLE - E(x,\phi)$ where $\phi = (\sigma,\lambda,\gamma)$ is a vector of parameters', the survival function $S(x,\phi)$, hazard function $h(x,\phi)$, inverse hazard function $\tau(x,\phi)$ and cumulative hazard function $H(x,\phi)$ for TLEE distribution are given by

$$S(x;\phi) = 1 - [1 - exp\{-2\lambda(exp\{\gamma x\} - 1)\}]^{\sigma} x \qquad \varepsilon R \quad (8)$$

$$h(x;\phi) = \frac{2\sigma\lambda\gamma\, exp\{-2\lambda(exp\{\gamma x\} - 1)\}\,[1 - exp\{-2\lambda(exp\{\gamma x\} - 1)\}]^{\sigma-1}}{exp\{-\gamma x\}\,(1 - [1 - exp\{-2\lambda(exp\{\gamma x\} - 1)\}]^{\sigma})} \qquad x \varepsilon R \quad (9)$$

$$\tau(x;\phi) = \frac{2\sigma\lambda\gamma\, exp\{-2\lambda(exp\{\gamma x\} - 1)\}}{exp\{-\gamma x\}\,[1 - exp\{-2\lambda(exp\{\gamma x\} - 1)\}]} \qquad x \varepsilon R \quad (10)$$

$$H(x;\phi) = -\,ln[1 - [1 - exp\{-2\lambda(exp\{\gamma x\} - 1)\}]^{\sigma}] \quad (11)$$

The densities graph of Topp Leone Exponential-exponential distribution is given below:

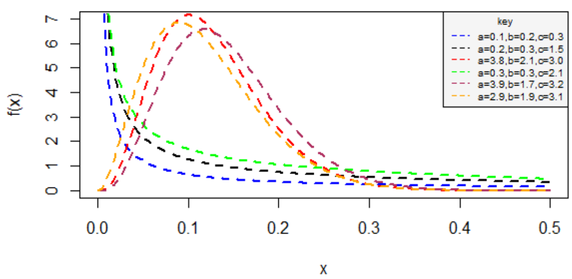

Figure 1: Plot of the TLEE density function for some parameters

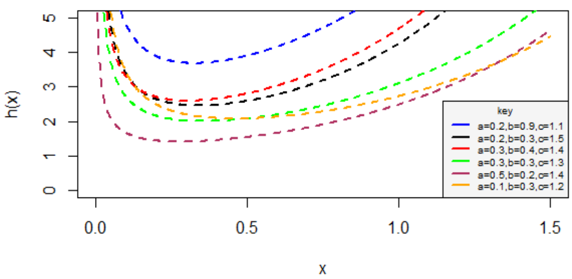

Figure 2: Plot of the TLEE hazard function for some parameters

3. LINEAR REPRESENTATION

This section introduced a useful linear representation for the TLEE pdf and cdf. Using generalized binomial and Taylor series expansions in equation (7), thus if $|x| \prec 1$ and $k \succ 0$ is a real non integer, the power series holds:

$$(1 - x)^{k-1} = \sum_{a=0}^{\infty} \frac{(-1)^a \Gamma(k)}{a!\,\Gamma(k-a)} x^a \quad (12)$$

Thus, applying the idea of equation (12) on the last term in (7), this becomes;

$$f_{TLE-E}(x;\sigma,\lambda,\gamma) = \frac{2\sigma\lambda\gamma}{exp\{-\gamma x\}} \sum_{a=1}^{\infty} (-1)^a \frac{\Gamma(\sigma)}{a!\,\Gamma(\sigma - a)} (exp\{-2\lambda(exp\{\gamma x\} - 1)\})^{a+1} \quad (13)$$

In equation (13) above, apply power series to the term:

$(exp\{-2\lambda(exp\{\gamma x\}-1)\})^{a+1} \Rightarrow \ [exp\{(-2\lambda(a+1)(exp\{\gamma x\}-1))\}]$ so equation (13) becomes:

$$f_{TLE-E}(x;\sigma,\lambda,\gamma)=\frac{2\sigma\lambda\gamma}{exp\{-\gamma x\}}\sum_{a,b=1}^{\infty}(-1)^{a+b}\frac{\Gamma(\sigma)(2\lambda)^b(a+1)^b}{a!\,b!\,\Gamma(\sigma-a)}(exp\{\gamma x\}-1)^b \tag{14}$$

Apply the idea of equation (12) on the last term in equation (14), this becomes:

$$f_{TLE-E}(x;\sigma,\lambda,\gamma)=\frac{2\sigma\lambda\gamma}{exp\{-\gamma x\}}\sum_{a,b,c=1}^{\infty}(-1)^{a+b+c}\frac{\Gamma(\sigma)(2\lambda)^b(a+1)^b b!\,(exp\{\gamma x\})^{b-c}}{a!\,b!\,c!\,(b-c)!\,\Gamma(\sigma-a)}$$

$$f_{TLE-E}(x;\sigma,\lambda,\gamma)=\sum_{a,b,c,d=1}^{\infty}(-1)^{a+b+c}\frac{2\sigma\lambda\Gamma(\sigma)(2\lambda)^b(a+1)^b b!(b-c)^d(\gamma x)^d}{a!b!c!d!(b-c)!\Gamma(\sigma-a)}\frac{\gamma}{exp\{-\gamma x\}} \tag{15}$$

$$f_{TLE-E}(x;\sigma,\lambda,\gamma)=\sum_{a,b,c,d=1}^{\infty}(-1)^{a+b+c}\frac{2\sigma\lambda\Gamma(\sigma)(2\lambda)^b(a+1)^b b!(b-c)^d(\gamma x)^d}{a!b!c!d!(b-c)!\Gamma(\sigma-a)}\gamma\big(1-(1-exp\{-\gamma x\})\big)^{-1} \tag{16}$$

Consider the idea of equation (17) below on the last of equation (16) above, this becomes:

$$(1-x)^{-1}=\sum_{b=0}^{\infty}\frac{(1+b-1)!}{b!\,(1-1)!}x^b \quad = \quad \sum_{b=0}^{\infty}x^b \qquad |x|<1, b>0 \tag{17}$$

$$f_{TLE-E}(x;\sigma,\lambda,\gamma)=\sum_{a,b,c,d,e=1}^{\infty}(-1)^{a+b+c}\frac{2\sigma\lambda\Gamma(\sigma)(2\lambda)^b(a+1)^b b!\,(b-c)^d(\gamma x)^d}{a!\,b!\,c!\,d!\,(b-c)!\,\Gamma(\sigma-a)}\gamma(1-exp\{-\gamma x\})^e$$

$$f_{TLE-E}(x;\sigma,\lambda,\gamma)=\sum_{a,b,c,d,e=1}^{\infty}(-1)^{a+b+c}\frac{2\sigma\lambda\Gamma(\sigma)(2\lambda)^b(a+1)^b b!\,(b-c)^d(\gamma x)^d}{a!\,b!\,c!\,d!\,(b-c)!\,\Gamma(\sigma-a)}\gamma(1-exp\{-\gamma x\})^e$$

$$f_{TLE-E}(x;\sigma,\lambda,\gamma)=\sum_{a,b,c,d,e,f=1}^{\infty}(-1)^{a+b+c+f}\frac{2\sigma\lambda\Gamma(\sigma)(2\lambda)^b(a+1)^b b!\,e!\,(b-c)^d(\gamma x)^d}{a!\,b!\,c!\,d!\,f!\,(b-c)!\,(e-f)!\,\Gamma(\sigma-a)}\gamma(exp\{-f\gamma x\})$$

$$f_{TLE-E}(x;\sigma,\lambda,\gamma)=\sum_{a,b,c,d,e,f=1}^{\infty}\eta_{a,b,c,d,f}\,\gamma(exp\{-f\gamma x\}) \tag{18}$$

Where:

$$\eta_{a,b,c,d,f}=(-1)^{a+b+c+f}\frac{2\sigma\lambda\Gamma(\sigma)(2\lambda)^b(a+1)^b b!\,e!\,(b-c)^d(\gamma x)^d}{a!\,b!\,c!\,d!\,f!\,(b-c)!\,(e-f)!\,\Gamma(\sigma-a)}$$

4. MATHEMATICAL PROPERTIES

This section provides some mathematical properties of the TLE-E distribution such as moments, and moment generating function, Rényi entropy and order statistics.

4.1 Moments and moment generating function

Suppose X is a random variable with TLE-E distribution, then the raw moment, say μ_n', is given by

$$\mu_n'=E\left(x^n\right)=\int_{-\infty}^{\infty}x^n f_{TLE-E}\left(x;\sigma,\lambda,\gamma\right)dx \tag{19}$$

$$=\sum_{a,b,c,d,e,f=1}^{\infty}\eta_{a,b,c,d,f}\int_{-\infty}^{\infty}x^n\gamma\exp\{-f\gamma x\}\left(x\right)dx \tag{20}$$

Thus:

$$E(x)=\sum_{a,b,c,d,e,f=1}^{\infty}\eta_{a,b,c,d,f}\left(\frac{1}{f\gamma}\right) and E(x^2)=\sum_{a,b,c,d,e,f=1}^{\infty}\eta_{a,b,c,d,f}\left(\frac{2}{f\gamma^2}\right)$$

Therefore, $Var(x) = \sum_{a,b,c,d,e,f=0}^{\infty}\eta_{a,b,c,d,f}\left(\frac{2}{f\gamma^2} - \left(\frac{1}{f\gamma}\right)^2\right) = \sum_{a,b,c,d,e,f=0}^{\infty}\eta_{a,b,c,d,f}\left(\frac{1}{f\gamma^2}\right)$ (21)

4.2 Renyi entropy

It plays an essential role in information theory. It has been used in applied statistics, queuing theory and reliability theory. It is used as indices of diversity and quantifies the uncertainty or randomness of a system. The Renyi Entropy for TLEE distribution is defined by

$$I_R(v) = (1-v)^{-1}\log\int_{-\infty}^{\infty} f^v(x)\,dx \qquad for\ v \succ 0\ \ and\ \ v \neq 1$$

From equation (7) above, thus

$$\left(f_{TLE-E}(x;\sigma,\lambda,\gamma)\right)^v = \frac{(2\sigma\lambda\gamma)^v}{exp\{-v\gamma x\}}(exp\{-2\lambda(exp\{\gamma x\}-1)\})^v[1-exp\{-2\lambda(exp\{\gamma x\}-1)\}]^{v(\sigma-1)}$$

$$\left(f_{TLE-E}(x;\sigma,\lambda,\gamma)\right)^v = \sum_{a,b,c,d,e,f=0}^{\infty}(-1)^{a+b+c+f}\frac{(2\sigma\lambda\gamma)^v\left(v(\sigma-1)\right)!b!e!(2\lambda)^b(a+v)^b(b-c)^d(\gamma x)^d}{a!b!c!d!f!(b-c)!(e-f)!\left(v(\sigma-1)-a\right)!}\gamma\left(\exp\{-f\gamma x\}\right)$$

$$\left(f_{TLE-E}(x;\sigma,\lambda,\gamma)\right)^v = \sum_{a,b,c,d,e,f=0}^{\infty}\psi_{a,b.c,d,f}\gamma(exp\{-f\gamma x\})$$

Where:

$$\psi_{a,b.c,d,f} = (-1)^{a+b+c+f}\frac{(2\sigma\lambda\gamma)^v\left(v(\sigma-1)\right)!\,b!\,e!\,(2\lambda)^b(a+v)^b(b-c)^d(\gamma x)^d}{a!\,b!\,c!\,d!\,f!\,(b-c)!\,(e-f)!\,(v(\sigma-1)-a)!}$$

Thus, the renyi entropy of TLEE distribution is defined as:

$$I_R(v) = (1-v)^{-1}\,log\int_{-\infty}^{\infty}\left(\left(f_{TLE-E}(x;\sigma,\lambda,\gamma)\right)^v = \sum_{a,b,c,d,e,f=0}^{\infty}\psi_{a,b.c,d,f}\gamma(exp\{-f\gamma x\})\right)dx \quad (22)$$

$$for v \succ 0 and v \neq 1$$

4.3 Order statistics

Order statistics plays a very significant role in statistics. Let $X_{1;n} \leq X_{2;n} \leq \ldots \leq X_{n;n}$ be the ordered sample from a continuous population with pdf f(x) and cdf F(x). The pdf of $X_{k;n}$, the kth order statistics is given by

$$f_{i;n} = \frac{f(x)}{\beta(i,n-i+1)}\sum_{j=0}^{n-i}(-1)^j\binom{n-1}{j}F^{j+i-1}(x)$$

From the equation, thus

$$f(x)F^{j+i-1}(x) = \frac{2\sigma\lambda\gamma}{exp\{-\gamma x\}}exp\{-2\lambda(exp\{\gamma x\}-1)\}\,[1-exp\{-2\lambda(exp\{\gamma x\}-1)\}]^{\sigma(j+i-1)+(\sigma-1)}$$

$$f(x)F^{j+i-1}(x) = \sum_{a,b,c,d,e,f=0}^{\infty}(-1)^{a+b+c+f}\frac{2\sigma\lambda\Gamma\left(\sigma(j+i)\right)b!\,(2\lambda)^b(a+1)^b(b-c)^d(\gamma x)^d}{a!\,b!\,c!\,d!\,f!\,(b-c)!\,(e-f)!\,\Gamma(\sigma(j+i)-a)}\gamma(exp\{-f\gamma x\})$$

$$f(x)F^{j+i-1}(x) = \sum_{a,b,c,d,e,f=0}^{\infty}\xi_{a,b,c,d,f}\,\gamma(exp\{-f\gamma x\})$$

Thus; the order statistics of TLEE distribution is defined as

$$f_{i;n} = \frac{f(x)}{\beta(i,n-i+1)}\sum_{j=0}^{n-i}(-1)^j\binom{n-1}{j}F^{j+i-1}(x) = \sum_{j=0}^{n-i}\sum_{a,b,c,d,e,f=0}^{\infty}\phi_j\xi_{a,b,c,d,f}\,\gamma(exp\{-f\gamma x\}) \quad (23)$$

Where:

$$\xi_{a,b,c,d,f} = (-1)^{a+b+c+f}\frac{2\sigma\lambda\Gamma\left(\sigma(j+i)\right)b!\,(2\lambda)^b(a+1)^b(b-c)^d(\gamma x)^d}{a!\,b!\,c!\,d!\,f!\,(b-c)!\,(e-f)!\,\Gamma(\sigma(j+i)-a)} and \phi_j = \frac{(-1)^j\binom{n-1}{j}}{\beta(i,n-i+1)}$$

5. PARAMETER ESTIMATION

Several approaches are used to estimate parameter, but the maximum likelihood method is the most commonly used among others. Therefore, the maximum likelihood estimators of the unknown parameters of the TLEE distribution from complete samples are determined. Let $X_1, \ldots, X_n$ be observed values from TLEE distribution with vector of parameters ϕ. The Log-likelihood function can be expressed as

$$l(\phi) = n\,log(2) + n\,log(\sigma) + n\,log(\lambda) + \sum_{i=1}^{n} log(\lambda\,exp\{-\gamma x\}) - 2\sum_{i=1}^{n} log(exp\{-\gamma x\}) - 2\lambda \sum_{i=1}^{n}(exp\{\gamma x\} - 1)$$
$$+(\theta - 1)\sum_{i=0}^{n} log(1 - exp\{-2\lambda(exp\{\gamma x\} - 1)\}) \qquad (24)$$

6. APPLICATION TO DATA SETS

We try to illustrate the flexibility of TLEE in application to real life data sets by comparing its performance with other existing distributions. The goodness-of-fit statistic and the MLE's for the models' parameters are presented in Tables1 and 2. To compare the fitted models, the paper used some goodness-of-fit measures which include Akaike information criterion (*AIC*), Bayesian information criterion (*BIC*).
We compared the fits of the new TLEE distribution with other competitive models distributions such as Burr X Exponential (BXE), Inverse Exponential (IE), Exponential Inverse Exponential (EIE), Exponential Exponential (EE), and Exponential (E) distributions. Their PDFs are available in literature.
Application 1: Modeling the survival times of 121 patient with breast cancer obtained from a large hospital in a period from 1929 – 1938 by Lee [13] and Ramos et al. [14].
0.3, 0.3, 4.0, 5.0, 5.6, 6.2, 6.3, 6.6, 6.8, 7.4, 7.5, 8.4, 8.4, 10.3, 11.0, 11.8, 12.2, 12.3, 13.5, 14.4, 14.4, 14.8, 15.5, 15.7, 16.2, 16.3, 16.5, 16.8, 17.2, 17.3, 17.5, 17.9, 19.8, 20.4, 20.9, 21.0, 21.0, 21.1, 23.0, 23.4, 23.6, 24.0, 24.0, 27.9, 28.2, 29.1, 30.0, 31.0, 31.0, 32.0, 35.0, 35.0, 37.0, 37.0, 37.0, 38.0, 38.0, 38.0, 39.0, 39.0, 40.0, 40.0, 40.0, 41.0, 41.0, 41.0, 42.0, 43.0, 43.0, 43.0, 44.0, 45.0, 45.0, 46.0, 46.0, 47.0, 48.0, 49.0, 51.0, 51.0, 51.0, 52.0, 54.0, 55.0, 56.0, 57.0, 58.0, 59.0, 60.0, 60.0, 60.0, 61.0, 62.0, 65.0, 65.0, 67.0, 67.0, 68.0, 69.0, 78.0, 80.0, 83.0, 88.0, 89.0, 90.0, 93.0, 96.0, 103.0, 105.0, 109.0, 109.0, 111.0, 115.0, 117.0, 125.0, 126.0, 127.0, 129.0, 129.0, 139.0, 154.0
Application 2: Modeling remission times (in months) of bladder cancer patients. This data set represents the remission times (in months) of a random sample of 128 bladder cancer patients by Lee and Wang [15]. This data is given as:
0.08, 5.85, 8.26, 11.98, 19.13, 1.76, 10.34, 14.83, 3.88, 5.32, 7.39, 3.25, 4.50,2.09, 3.48, 4.87, 0.81, 2.62,11.64, 17.36, 1.40, 3.02, 4.34, 34.26, 0.90 , 3.52, 4.98, 6.97, 9.02, 13.29, 0.40, 2.26,3.57, 5.06, 7.09, 9.22, 13.80, 25.74, 0.50, 2.46, 3.64, 5.09, 7.26, 9.47, 14.24, 25.82, 0.51, 2.54, 3.70, 5.17, 5.41,7.62, 10.75, 16.62, 43.01, 1.19, 2.75, 4.33, 5.49, 7.66, 11.25, 4.26, 5.41, 7.63, 17.12, 46.12, 1.26, 2.83, 17.14, 79.05, 1.35, 2.87, 12.07, 21.73, 2.07, 3.36, 6.93,5.62, 7.87, 3.82, 6.94, 8.66, 13.11, 23.63, 0.20, 2.23, 14.77, 32.15, 2.64, 5.71, 7.28, 9.74, 14.76, 26.31, 5.32, 7.32, 10.06, 2.69, 4.18, 5.34, 7.59, 10.66, 15.96, 36.66, 1.05, 2.69,4.23, 7.93, 11.79, 18.10, 1.46, 4.40, 6.25, 8.37, 12.02, 2.02, 3.31, 4.51, 6.54, 8.53, 12.03, 20.28, 2.02, 3.36, 6.76, 8.65, 12.63, 22.69.
The R codes are used to compute and evaluate MLEs and log – likelihood function. The measure of goodness of fit used are: AIC and BIC, the lower the values of the criteria the better the fit. The value of TLEE distribution is compared with those of BXE, EIE, IE and E. Some goodness of fit criteria and MLEs of the models for the first data set and second data set are presented in Tables (1) and (2), respectively.
For the first and second real data sets, the values in Tables 1 & 2 indicate that, the TLEE model has the lowest values for AIC and BIC among the fitted models (BXE, EIE, EE, IE, and E).
The pdf's and hazard function's plots are displayed in Figures 1 & 2 respectively. It is clear from this two Figures that the densities of the new distributions (TLEE) is inverse J shape and skewed to the right with flat tail, while its hazard functions' (rate) shape exhibits increasing shape that is upward bathtub shape.

Table 1. MLEs with their standard errors (in parentheses) and some Goodness-of-fit test for the breast cancer data

Models	Estimate			$-2\hat{\ell}$	AIC	BIC
TLEE	σ =1.247	λ =1.709	$\gamma = 0.006$	1157.5042	1163.504	1161.670
E	γ =0.0216			1170.2554	1172.255	1172.3382
EIE	$\lambda = 0.3510$	$\gamma = 0.0076$		1169.8026	1173.803	1173.9682

IE	γ =10.3215		1354.5582	1356.558	1356.641
BXE	φ =0.1330	γ =0.0007	1430.9646	1434.965	1435.1302

Table 2. MLEs with their standard errors (in parentheses) and the Goodness-of-fit criteria for the bladder cancer patients' data

Models	Estimates			$-2\hat{\ell}$	AIC	BIC
TLEE	σ =1.213	λ =186.5	γ = 0.0003	826.339	832.339	830.5534
EE	λ =198.8	γ =0.0005		828.8492	832.8492	833.0636
IE	γ =2345			920.7656	922.7646	922.8782
BXE	φ =0.1291	γ =0.0033		1085.6258	1089.6258	1089.8402

7. SUMMARY AND CONCLUSIONS

A new continuous distribution called Topp Leone Exponential - Exponential (TLEE) with three parameters is developed. Some mathematical properties of this new distribution were derived. The flexibility and usefulness of one of the new distributions by studying the remission times (in months) of some random sample of 128 bladder cancer patients and 63 observations of the strengths of 1.5 cm glass fibers obtained by workers at the UK National Physical Laboratory study is demonstrated. It is shown that; the128 bladder cancer patients and the 63 observations of the strengths of 1.5 cm glass fibers sets of data can be modeled using TLEE distribution.

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