POLYNOMIAL SEQUENCE AND EXPONENTIAL GENERATING FUNCTION OF LUCAS SERIES

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ABSTRACT: In teaching mathematics at high schools, the mathematics competency of teacher is an important procedure in teaching process. In this paper we study the properties of polynomial sequence and exponential generating function of Lucas series and I pointed out some application in mathematics. And i hope it will help the teachers to develop high school their mathematics competency in Vietnam.

KEYWORDS: Teachers' mathematics competency, Polynomial sequence, Exponential generating functions, Lucas series.

1. EXPONENTIAL GENERATING FUNCTION OF POLYNOMIAL SEQUENCE $(a_n(x))$

1.1. Formula terms of polynomial sequence $(a_n(x))$

Consider the polynomial sequences $(a_n(x))$ such that

(1) $a_0(x) = p_0, a_1(x) = p_1$
(2) $a_{n+2}(x) = pa_{n+1}(x) + qa_n(x)$ where $n \geq 0$
(3) $p_0, p_1, p, q \in \mathbb{R}[x]$.

Problems: Determining the polynomial $a_n(x)$ and its properties.

Theorem 1. Let $\Delta = p^2 + 4q$ and $t_1 = \frac{-p-\sqrt{\Delta}}{2}, t_2 = \frac{-p+\sqrt{\Delta}}{2}$, we have

$$a_n(x) = \frac{1}{\sqrt{\Delta}}[(\frac{p+\sqrt{\Delta}}{2})^n(p_1 + t_2p_0) - (\frac{p-\sqrt{\Delta}}{2})^n(p_1 + t_1p_0)].$$

Proof. Consider the sequences $(b_n(x))$ where $b_1(x) = a_0(x) = p_0$ and $b_{n+1}(x) = a_n(x)$ for all $n \geq 1$. Deduce we have two polynomial sequences:

$$a_1(x) = p_1, b_1(x) = p_0$$
$$a_{n+1}(x) = pa_n(x) + qb_n(x)$$
$$b_{n+1}(x) = a_n(x), n \geq 1$$

By changing
$a_{n+1}(x) + tb_{n+1}(x) = (p + t)a_n(x) + qb_n(x)$.

Choosing t such that $t(t + p) = q$.
Then $t_1 = \frac{-p-\sqrt{\Delta}}{2}$ and $t_2 = \frac{-p+\sqrt{\Delta}}{2}$.
Set $t = t_i$ deduce

$$a_{n+1}(x) + ta_n(x) = (p + t)^n(a_1(x) + tb_1(x)).$$

Choosing $t = t_2$ and $t = t_1$, we have

$$\begin{cases} a_{n+1}(x) + t_2a_n(x) = (\frac{p+\sqrt{\Delta}}{2})^n(p_1 + t_2p_0) \\ a_{n+1}(x) + t_1a_n(x) = (\frac{p-\sqrt{\Delta}}{2})^n(p_1 + t_1p_0). \end{cases}$$

We get

$$a_n(x) = \frac{1}{\sqrt{\Delta}}[(\frac{p+\sqrt{\Delta}}{2})^n(p_1 + t_2p_0) - (\frac{p-\sqrt{\Delta}}{2})^n(p_1 + t_1p_0)].$$

Remark 1. We consider $\sqrt{\Delta}$ in $\mathbb{C}$ and the above result is always there when $x \in \mathbb{R}$.

Theorem 2. We have

$$a_{n+1}^2 - a_na_{n+2} = (-q)^n(p_1^2 - pp_0p_1 - qp_0^2).$$

Proof. By changing

$$\begin{pmatrix} a_{n+1} & a_n \\ a_{n+2} & a_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ q & p \end{pmatrix} \begin{pmatrix} a_n & a_{n-1} \\ a_{n+1} & a_n \end{pmatrix}.$$

we have

$$\begin{pmatrix} a_{n+1} & a_n \\ a_{n+2} & a_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ q & p \end{pmatrix}^n \begin{pmatrix} a_1 & a_0 \\ a_2 & a_1 \end{pmatrix}.$$

The determinant for both sides

$a_{n+1}^2 - a_n a_{n+2} = (-q)^n (p_1^2 - pp_0 p_1 - qp_0^2).$

Corollary 1. With $p_1 = \frac{pp_0}{2}$ we have

$$a_n(x) = \frac{p_0}{2}[(\frac{p+\sqrt{\Delta}}{2})^n + (\frac{p-\sqrt{\Delta}}{2})^n].$$

1.2. Exponential generating function of polynomial sequence $(a_n(x))$

Set $u = \frac{p+\sqrt{\Delta}}{2}, v = \frac{p-\sqrt{\Delta}}{2}$ and $f = \frac{p_1+t_2p_0}{\sqrt{\Delta}}, g = \frac{p_1+t_1p_0}{\sqrt{\Delta}}$.
Consider the polynomial sequence $(a_n(x))$ is

$$A(x,t) = \sum_{n=0}^{\infty} a_n(x)\frac{t^n}{n!}.$$

Set

$$A(x,t) = \sum_{n=0}^{\infty}(fu^n - gv^n)\frac{t^n}{n!}$$
$$= f\sum_{n=0}^{\infty}\frac{u^nt^n}{n!} - g\sum_{n=0}^{\infty}\frac{v^nt^n}{n!}$$
$$= fe^{ut} - ge^{vt}$$
$$= e^{vt}(fe^{(u-v)t} - g)$$
$$= fe^{vt}(e^{\sqrt{\Delta}t} - g/f).$$

Let $p_0 = 0, p_1 \neq 0$.
We get $f = g \neq 0$ and

$$A(x,t) = fe^{vt}(e^{\sqrt{\Delta}t} - 1) = \frac{p_1}{\sqrt{\Delta}}e^{vt}(e^{\sqrt{\Delta}t} - 1).$$

Proposition 1. We have

$$A(x,t) = \frac{p_1}{\sqrt{\Delta}}e^{vt}(e^{\sqrt{\Delta}t} - 1).$$

Lemma 1.
We have $\frac{zt}{e^{zt}-1} = 1 + \frac{\beta_1 zt}{1!} + \frac{\beta_2 z^2t^2}{2!} + \cdots + \frac{\beta_n z^nt^n}{n!} + \cdots$ where $\beta_{2n+1} = 0, n = 1,2,3, \ldots$.
Proof. From

$$e^{zt} - 1 = zt(1 + \frac{zt}{2!} + \frac{z^2t^2}{3!} + \cdots + \frac{z^nt^n}{(n+1)!} + \cdots)$$

where $z = \sqrt{\Delta}$, we deduce

$$\frac{zt}{e^{zt}-1} = \frac{1}{1 + \frac{zt}{2!} + \frac{z^2t^2}{3!} + \cdots + \frac{z^nt^n}{(n+1)!} + \cdots}.$$

Description

$$\frac{zt}{e^{zt}-1} = 1 + \frac{\beta_1 zt}{1!} + \frac{\beta_2 z^2t^2}{2!} + \cdots + \frac{\beta_n z^nt^n}{n!} + \cdots.$$

We have

$$1 = (1 + \frac{\beta_1 zt}{1!} + \cdots + \frac{\beta_n z^nt^n}{n!} + \cdots)$$
$$(1 + \frac{zt}{2!} + \frac{z^2t^2}{3!} + \cdots + \frac{z^nt^n}{(n+1)!} + \cdots).$$

From thence is inferred

$$\frac{\beta_n}{n!1!} + \frac{\beta_{n-1}}{(n-1)!2!} + \cdots + \frac{\beta_1}{1!n!} + \frac{1}{(n+1)!} = 0$$

forall $n = 1,2, \ldots$, or

$$\binom{n+1}{1}\beta_n + \binom{n+1}{2}\beta_{n-1} + \cdots$$
$$+ \binom{n+1}{n}\beta_1 + \binom{n+1}{n+1} = 0$$

and we have

$$\beta_1 = -\frac{1}{2}, \beta_2 = \frac{1}{6}, \beta_3 = 0, \beta_4 = -\frac{1}{30}, \beta_5 = 0.$$

Thus, we change function by a formal power series:

$$\frac{zt}{e^{zt}-1} = 1 + \frac{\beta_1 zt}{1!} + \frac{\beta_2 z^2t^2}{2!} + \cdots + \frac{\beta_n z^nt^n}{n!} + \cdots.$$

Note that, $\frac{zt}{e^{zt}-1} + \frac{zt}{2} = \frac{zt}{2}.\frac{e^{zt}+1}{e^{zt}-1} = \frac{zt}{2}.\coth\frac{zt}{2}$.
Because the function $\frac{zt}{2}.\coth\frac{zt}{2}$ is even, thus we deduce $\beta_{2n+1} = 0.\ n = 1,2,3, \ldots$.

Corollary 2. We have

$$\pi x coth \pi x = 1 + 2\sum_{n=1}^{\infty}(-1)^{n-1}s_{2n}x^{2n}$$

and

$$(-1)^{n-1}\beta_{2n} = \frac{2(2n!)}{(2\pi)^{2n}}s_{2n}, n = 1,2,3, \ldots.$$

Proof. From
Same $\frac{2x}{e^{2x}-1} + x$, and

$$xcoth\pi x = 1 + 2\sum_{n=1}^{\infty}(-1)^{n-1} s_{2n}x^{2n}$$

we deduce $\frac{\beta_{2n}2^{2n}\pi^{2n}}{(2n)!} = 2(-1)^{n-1}s_{2n}$ or $(-1)^{n-1}\beta_{2n} = \frac{2(2n!)}{(2\pi)^{2n}}s_{2n}$.

Remark 2. $B_n = (-1)^{n-1}\beta_{2n}$is nth Bernoulli number.

Proposition 2. We have

$$\sum_{k=0}^{n+1}\binom{n+1}{k} a_{n+1-k}(x)\beta_k\sqrt{\Delta^k} = p_1\binom{n+1}{n}(\tfrac{p-\sqrt{\Delta}}{2})^n,$$

where $n = 1,2,3,\ldots$ and $\beta_0 = 1$.

Proof. From $A(x,t)\frac{\sqrt{\Delta}t}{e^{\sqrt{\Delta}t}-1} = p_1te^{vt}$ we have

$$p_1te^{vt} = (\sum_{n=0}^{\infty}a_n(x)\frac{t^n}{n!})(1+\frac{\beta_1\sqrt{\Delta}t}{1!}+\cdots + \frac{\beta_n\sqrt{\Delta^n}t^n}{n!}+\cdots)$$

And

$$p_1e^{vt} = (\textstyle\sum_{n=1}^{\infty}a_n(x)\frac{t^{n-1}}{n!})(1+\frac{\beta_1\sqrt{\Delta}t}{1!}+\cdots+\frac{\beta_n\sqrt{\Delta^n}t^n}{n!}+\cdots).$$

Now we compare coefficients t^n for both sides, we have

$$\frac{p_1v^n}{n!} = \frac{a_{n+1}(x)}{(n+1)!}+\frac{a_n(x)\beta_1\sqrt{\Delta}}{n!\,1!}+\cdots + \frac{a_2(x)\beta_{n-1}\sqrt{\Delta^{n-1}}}{2!\,(n-1)!} + \frac{a_1(x)\beta_n\sqrt{\Delta^n}}{1!\,n!}$$

or

$$\sum_{k=0}^{n+1}\binom{n+1}{k} a_{n+1-k}(x)\beta_k\sqrt{\Delta^k} = p_1\binom{n+1}{n}(\tfrac{p-\sqrt{\Delta}}{2})^n$$

where $n = 1,2,3,\ldots$

Proposition 3. Suppose

$$\frac{\sqrt{\Delta}t}{e^{\sqrt{\Delta}t}-1} = 1 + \sum_{k=1}^{\infty}\frac{g_k\sqrt{\Delta^k}t^k}{k!}.$$

Then we have

$$\binom{n+1}{0}f_{n+1}+\binom{n+1}{1}f_ng_1\sqrt{\Delta} + \binom{n+1}{2}f_{n-1}g_2\sqrt{\Delta^2}+ \cdots + \binom{n+1}{n}f_1g_n\sqrt{\Delta^n} = (n+1)v^n.$$

2. EXPONENTIAL GENERATING FUNCTION OF FIBONACCI AND LUCAS POLYNOMIAL SEQUENCE

2.1. Some results about Fibonacci (f_n) polynomial sequence

Example 1. With polynomial sequence $f_0 = 0, f_1 = 1$ and $f_{n+2} = pf_{n+1} + qf_n$ we have

$$f_n = \frac{1}{\sqrt{\Delta}}[(\frac{p+\sqrt{\Delta}}{2})^n - (\frac{p-\sqrt{\Delta}}{2})^n]$$

and $f_n^2 - f_{n-1}f_{n+1} = (-q)^{n-1}$.

Proof. We have $f_n = \frac{p}{\sqrt{\Delta}}((\frac{p+\sqrt{\Delta}}{2})^n - (\frac{p-\sqrt{\Delta}}{2}))$ and $\begin{pmatrix} f_n & f_{n-1}\\ f_{n+1} & f_n\end{pmatrix} = \begin{pmatrix} 0 & 1\\ q & p\end{pmatrix}\begin{pmatrix} f_{n-1} & f_{n-2}\\ f_n & f_{n-1}\end{pmatrix}$, then we get $f_n^2 - f_{n-1}f_{n+1} = (-q)^{n-1}$.

Example 2. Given polynomial sequence $f_0 = 0, f_1 = 1$ and $f_{n+2} = pf_{n+1} + qf_n$ where $n \geq 0$. Determination of q by p such that

$$f_n^4 = f_{n-2}f_{n-1}f_{n+1}f_{n+2} + p^{4n-4}.$$

Proof. We have $f_n^2 - f_{n-1}f_{n+1} = (-q)^{n-1}$. Changing $T = f_{n-2}f_{n+2}$ by

$$T = \frac{f_n - pf_{n-1}}{q}.(pf_{n+1}qf_n)$$
$$= f_n^2 - \frac{p^2}{q}f_{n-1}f_{n+1} + \frac{p}{q}f_nf_{n+1} - pf_{n-1}f_n.$$

therefore $f_{n-2}f_{n+2} = f_n^2 - p^2(-q)^{n-2}$. From two systems formulas, we get

$$f_{n-2}f_{n-1}f_{n+1}f_{n+2} = [f_n^2 - (-q)^{n-1}][f_n^2 - p^2(-q)^{n-3}].$$

therefore $f_n^4 = f_{n-2}f_{n-1}f_{n+1}f_{n+2} + p^{4n-4}$ if and only if $q = p^2 > 0$.

Example 3. With Fibonacci polynomial sequence (f_n) we have $\sum_{k=0}^{n}\binom{n}{k}q^{n-k}p^kf_k = f_{2n}$.

Proof. Set $\alpha = \frac{p+\sqrt{\Delta}}{2}$ and $\beta = \frac{p-\sqrt{\Delta}}{2}$.When α, β are two solutions of $t^2 - pt - q = 0$ and deduce $q + p\alpha = \alpha^2, q + p\beta = \beta^2$. Total transformation

$$T = \sum_{k=0}^{n} \binom{n}{k} q^{n-k} p^k f_k$$
$$= \frac{1}{\sqrt{\Delta}} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} p^k (\alpha^k - \beta^k)$$
$$= \frac{1}{\sqrt{\Delta}} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} p^k \alpha^k - \frac{1}{\sqrt{\Delta}} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} p^k \beta^k$$
$$= \frac{1}{\sqrt{\Delta}} (^q - \frac{1}{\sqrt{\Delta}} (^q$$
$$= f_{2n}.$$

From this result we deduce

$$\sum_{k=0}^{n} \binom{n}{k} q^{n-k} p^k f_k = f_{2n}.$$

2.2. Some results about Lucas Polynomial series (l_n)

Lemma 2. With Lucas polynomial sequence $l_0 = 2, l_1 = p$ and $l_{n+2} = pl_{n+1} + ql_n$ when integer $n \geq 0$ we have polynomial $l_n = (\frac{p+\sqrt{\Delta}}{2})^n + (\frac{p-\sqrt{\Delta}}{2})^n$.
Theorem 3. we always have

$$p^n l_n = q^n \sum_{k=1}^{n} (-1)^{n-k} \binom{n}{k} (\frac{l_{2k}}{q^k} - 2).$$

Proof. Set $\alpha = \frac{p+\sqrt{\Delta}}{2}$ and $\beta = \frac{p-\sqrt{\Delta}}{2}$.
Total transformation

$$\sum_{k=0}^{n} \binom{n}{k} q^{n-k} p^k l_k = \sum_{k=0}^{n} \binom{n}{k} q^{n-k} p^k (\alpha^k + \beta^k)$$
$$= (q + p\alpha)^n + (q + p\beta)^n$$

and deduce

$$\sum\nolimits_{k=0}^{n} \binom{n}{k} q^{n-k} p^k l_k = l_{2n}$$
Or
$$\sum_{k=0}^{n} \binom{n}{k} (\frac{p}{q})^k l_k = \frac{l_{2n}}{q^n}.$$

Thence inferred that

$$\sum_{k=1}^{n} \binom{n}{k} (\frac{p}{q})^k l_k = \frac{l_{2n}}{q^n} - 2.$$

use the discrete inverse-transform method we get

$$(\frac{p}{q})^n l_n = \sum_{k=1}^{n} (-1)^{n-k} \binom{n}{k} (\frac{l_{2k}}{q^k} - 2)$$
Or
$$p^n l_n = q^n \sum_{k=1}^{n} (-1)^{n-k} \binom{n}{k} (\frac{l_{2k}}{q^k} - 2).$$

Theorem 4. Suppose $\frac{1}{e^{\sqrt{\Delta}t}+1} = \sum\nolimits_{n=0}^{\infty} \frac{h_n \sqrt{\Delta^n} t^n}{n!}$.
Then we have $\beta^n = \sum\nolimits_{k=0}^{n} \binom{n}{k} h_k l_{n-k} \sqrt{\Delta^k}$.
Proof. Let exponential generating functions

$$l(x,t) = \sum_{n=0}^{\infty} \frac{l_n t^n}{n!}.$$

Set $\alpha = \frac{p+\sqrt{\Delta}}{2}$ and $\beta = \frac{p-\sqrt{\Delta}}{2}$.
Then

$$l(x,t) = \sum_{n=0}^{\infty} \frac{(\alpha^n + \beta^n) t^n}{n!} = e^{\alpha t} + e^{\beta t}$$

and deduce
$$e^{\beta t} = l(x,t) \frac{1}{e^{\sqrt{\Delta}t}+1} =$$
$$(\sum\nolimits_{n=0}^{\infty} \frac{l_n t^n}{n!}) . (\sum\nolimits_{n=0}^{\infty} \frac{h_n \sqrt{\Delta^n} t^n}{n!}).$$

We compare coefficients t^n we get

$$\beta^n = \sum_{k=0}^{n} \binom{n}{k} h_k l_{n-k} \sqrt{\Delta^k}.$$

Remark 3. Because $\frac{1}{e^{\sqrt{\Delta}t}+1} - \frac{1}{2} = \frac{1}{2} \frac{1-e^{\sqrt{\Delta}t}}{e^{\sqrt{\Delta}t}+1}$ is odd function, therefore $h_{2n} = 0$ where $n \geq 1$.

2.3. Some examples of application

According to [5][3][4] and [1], we have some examples of applications.
Example 4. Given polynomial sequence $l_0 = 2, l_1 = 1$ and $l_{n+2} = pl_{n+1} + ql_n$ where $n \geq 0$. Prove that, $l_n^4 = l_{n-2} l_{n-1} l_{n+1} l_{n+2} + \Delta^2 p^{4n-4}$ if and only if $q = p^2$.

Example 5. [1] Given polynomial sequences (f_n) such that $f_0 = 2, f_1 = 3x$ and $f_{n+2} = 3xf_{n+1} + (1 - x - 2x^2)f_n, n \geq 0$. Find all positive integers n such that f_n divisible by $x^3 - x^2 + x$.
Proof. According to the theorem 1, we have $f_n(x) = (2x-1)^n + (x+1)^n$. Therefore $f_n(x)$ divisible by x if and only if $f_n(0) = 0$ or $(-1)^n + 1 = 0$. We get n is odd number. We notation u is complex solution of $x^2 - x + 1$. Because $u^2 + u = 2u - 1$ then $f_n(x)$ divisible by $x^2 - x + 1$ if and only if $(2u-1)^n + (u+1)^n = 0$ or $(u^2+u)^n + (u+1)^n = 0$. Because $u + 1 \neq 0$ therefore $u^n + 1 = 0$. Because $u = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3}$ therefore $u^n + 1 = 0$ is equivalent to $\cos\frac{n\pi}{3} + i\sin\frac{n\pi}{3} = -1$. we get $n = 6m + 3$. Because we have $f_{6m+3}(x)$ divisible by $x^3 - x^2 + x$ where $m \in \mathbb{N}$.

Example 6. [3][4] Given the polynomial sequences (f_n) must satisfy requirements $f_0 = 2, f_1 = 1$ and $f_{n+2} = f_{n+1} + (x^2 + x)f_n, n \geq 0$. Find all positive integers n such that $f_n + 1$ divisible by $x^2 + x + 1$.
Proof. According to the theorem 1, we have $f_n(x) = (x+1)^n + (-x)^n + 1$. Therefore $f_n(x) + 1$ divisible by $x^2 + x + 1$ if and only if $f_n(u) = 0$, where u is integer solution of $x^3 - 1 = 0$ or u is solution of $x^2 + x + 1$. Because $u^3 = 1$ therefore $n = 6k + s$ where k is integer number, $s = 0,1,2,3,4,5$ we have $f_n(u) + 1 = 0$ is equivalent to $(u+1)^s + (-u)^s + 1 = 0$. Therefore we get $s = 2$ and $s = 4$.

Example 7. Given the polynomial sequences $\{f_n\}$ must satisfy requirements $f_0 = 3$, $f_1 = 2x + 2$, $f_2 = 2x^2 + 2x + 2$ and $f_{n+3} = 2(x+1)f_{n+2} - (x^2 + 3x + 1)f_{n+1} + (x^2 + x)f_n, n \geq 0$. Find all natural number m such that f_m divisible by $(x^2 + x + 1)^2$.
Proof. The characteristic equations $t^3 - 2(x+1)t^2 + (x^2 + 3x + 1)t - (x^2 + x) = 0$ are three solutions $t_1 = x + 1, t_2 = x$ and $t_3 = 1$. According to inductive method by n we have $f_n(x) = (x+1)^n + x^n + 1$. Polynomial $x^2 + x + 1$ is irreducible in $\mathbb{Q}$ with complex solution $\alpha = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}, \alpha^3 = 1$. Because $\alpha^2 + \alpha + 1 = 0$ deduce $1 + \alpha = -\alpha^2$. Therefore $(1+\alpha)^6 = 1$. Set $m = 6k + r$ where $r \in \{0,1,2,3,4,5\}$. Polynomial $(x+1)^m + x^m + 1$ divisible by $(x^2 + x + 1)^2$ if and only if $\begin{cases} (1+\alpha)^m + \alpha^m + 1 = 0 \\ m(1+\alpha)^{m-1} + m\alpha^{m-1} = 0 \end{cases}$ is equivalent to $\begin{cases} r = 2, r = 4 \\ (1+\alpha)^{r-1} + \alpha^{r-1} = 0. \end{cases}$ we realize that $r = 4$ is contentment. Therefore, polynomial $(x+1)^m + x^m + 1$ divisible by $(x^2 + x + 1)^2$ if and only if $m = 6k + 4$.

Example 8. Given sequence $a_1 = 1, a_n = -1a_{n-1} + 2a_{n-2} - \cdots + (-1)^{n-1}(n-1)a_1$ forall integer $n \geq 2$. Then we have $\sum_{k=1}^{2n} C_{2n}^k a_{k+1} = F_{2n-1}$..
Proof. Set $f(x) = a_1x + a_2x^2 + a_3x^3 + \cdots$. Multiply two exponent series

$$\begin{aligned} F(x) &= f(x)(-1x + 2x^2 - 3x^3 + \cdots) \\ &= -1a_1x^2 + (-1a_2 + 2a_1)x^3 \\ &\quad + (-1a_3 + 2a_2 - 3a_1)x^4 + \cdots \\ &= a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + \cdots \\ &= f(x) - x \end{aligned}$$

From $\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - x^5 + \cdots$ deduce exponent series.
Therefore we get $\frac{-x}{(1+x)^2} = -1x + 2x^2 - 3x^3 + 4x^4 - 5x^5 + \cdots$.
Then $F(x)$ has

$$f(x)\left(\frac{-x}{(1+x)^2}\right) = f(x) - x$$

Or

$$f(x)[x^2 + 3x + 1] = x^3 + 2x^2 + x.$$

From $[a_1x + a_2x^2 + a_3x^3 + \cdots][x^2 + 3x + 1] = x^3 + 2x^2 + x$ we deduce

$$\begin{aligned} &a_1 = 1, a_2 + 3a_1 = 2, a_3 + 3a_2 = 0, \\ &a_{n+2} + 3a_{n+1} + a_n = 0, n \geq 2. \end{aligned}$$

From a closed-form $f = \frac{x^3 + 2x^2 + x}{x^2 + 3x + 1} = x - 1 + \frac{3x+1}{x^2 + 3x + 1}$ we have

$$\begin{aligned} f &= x - 1 + \frac{3x + 1}{(x + \frac{3+\sqrt{5}}{2})(x + \frac{3-\sqrt{5}}{2})} \\ &= x - 1 + \frac{1}{\sqrt{5}}\left(\frac{u^2}{x-u} - \frac{v^2}{x-v}\right) \end{aligned}$$

where $u = \frac{-3-\sqrt{5}}{2}, v = \frac{-3+\sqrt{5}}{2}$.
Therefore $f = x - 1 + \frac{1}{\sqrt{5}}\left(\frac{u}{vx-1} - \frac{v}{ux-1}\right)$ because $uv = 1$, and we get $f = x - 1 + \frac{1}{\sqrt{5}}\left(\frac{v}{1-ux} - \frac{u}{1-vx}\right)$.
Then we have

$$\begin{aligned} f &= x - 1 + \frac{v}{\sqrt{5}}(1 + ux + u^2x^2 + u^3x^3 + \cdots) \\ &\quad - \frac{u}{\sqrt{5}}(1 + vx + v^2x^2 + v^3x^3 + \cdots) \end{aligned}$$

and we've seen the formula for determining

$$a_n = \frac{u^{n-1} - v^{n-1}}{\sqrt{5}}$$

or

$$a_n = \frac{u^{n-1} - v^{n-1}}{\sqrt{5}}$$
$$= (-\frac{(\frac{1+\sqrt{5}}{2})^{2n-2} - (\frac{1-\sqrt{5}}{2})^{2n-2}}{\sqrt{5}}.$$

Deduce $a_n = (-1)^{n-1} F_{2n-1}$ for all $n \geq 2$. From

$$\sum_{k=1}^{2n} C_{2n}^k a_{k+1} = \frac{1}{\sqrt{5}} \sum_{k=1}^{2n} C_{2n}^k [u^k - v^k]$$
$$= \frac{(1+u)^{2n} - (1+v)^{2n}}{\sqrt{5}}$$

we get $\sum_{k=1}^{2n} C_{2n}^k a_{k+1} = \frac{(\frac{1+\sqrt{5}}{2})^{2n} - (\frac{1-\sqrt{5}}{2})^{2n}}{\sqrt{5}} = F_{2n-1}$.

Example 9. Given sequence (a_n) must satisfy requirements $a_1 = 1$ and $a_n = 1a_{n-1} + 2a_{n-2} + \cdots + (n-1)a_1$ forall integer $n \geq 2$. Prove that $a_n = F_{2n-1}$, where $F_0 = F_1 = 1, F_{n+2} = F_{n+1} + F_n, n \geq 0$.

Proof. Consider $f(x) = a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots$. Then we have $f(x)(1x + 2x^2 + \cdots + nx^n + \cdots) = f(x) - x$. From $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$ we deduce $1x + 2x^2 + \cdots = \frac{x}{(x-1)^2}$.

Therefore $f(x) = x + \frac{x^2}{x^2 - 3x + 1}$ and $f(x)(x^2 - 3x + 1) = x^3 - 2x^2 + x$.

Now we compare coefficients $x^n, n \geq 1$ for both sides, we get $a_3 = 3a_2$, and $a_{n+2} - 3a_{n+1} + a_n = 0$ forall integer $n \geq 2$.

From closed-form $f(x) = \frac{x^3 - 2x^2 + x}{x^2 - 3x + 1} = x + 1 + \frac{3x-1}{x^2 - 3x + 1}$ we have

$$f(x) = x + 1 + \frac{3x - 1}{(x - \frac{3+\sqrt{5}}{2})(x - \frac{3-\sqrt{5}}{2})}$$
$$= x + 1 + \frac{1}{\sqrt{5}}(\frac{a^2}{x-a} - \frac{b^2}{x-b})$$

where $a = \frac{3+\sqrt{5}}{2}, b = \frac{3-\sqrt{5}}{2}$.

We have $f(x) = x + 1 + \frac{1}{\sqrt{5}}(\frac{a}{bx-1} - \frac{b}{ax-1})$ because $ab = 1$. Therefore $f(x) = x + 1 + \frac{1}{\sqrt{5}}(\frac{b}{1-ax} - \frac{a}{1-bx})$.

Then we have sequences

$$f(x) = x + 1 + \frac{b}{\sqrt{5}}(1 + ax + a^2x^2 + a^3x^3 + \cdots)$$
$$- \frac{a}{\sqrt{5}}(1 + bx + b^2x^2 + b^3x^3 + \cdots)$$

and we've seen the formula for determining $a_n = \frac{a^{n-1} - b^{n-1}}{\sqrt{5}}, n \geq 2$.

We have

$$a_n = \frac{a^{n-1} - b^{n-1}}{\sqrt{5}}$$
$$= \frac{(\frac{1+\sqrt{5}}{2})^{2n-2} - (\frac{1-\sqrt{5}}{2})^{2n-2}}{\sqrt{5}} = F_{2n-1}.$$

3. CONCLUSIONS

We need to be on helping teachers use polynomial sequence and exponential generating function of Lucas series as part of teaching and learning, in ways that will raise pupil's achievement. The effectiveness of teaching is not confirmed through what students learn by passing on but by what students can do. And i hope it will help the teachers to develop high school their mathematics competency in Vietnam.

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