COMPUTING FOURIER INTEGRAL USING MATLAB

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ABSTRACT: This work studies the computation of Fourier integral using Matlab software. The expressions obtained were computed in Matlab using ezplot command. The plots in line and contour form were presented. From the outcome of the result it was observed that the functions increase with increasing values of π.
KEYWORDS: Fourier integral, Matlab

1. INTRODUCTION

Fourier transform introduce by French mathematician, Fourier Jean Baptist called Joseph Fourier in 18th century. The Fourier transform named for Joseph Fourier is a mathematical transform with many applications in physics and engineering (Fourier, 1822). Very commonly, it expresses a mathematical function of time as a function of frequency known as its frequency spectrum. The Fourier inversion theorems detail this relationship for instance, the transform of a musical chord made up of pure note (without overtones) expressed as amplitude as a function of time is a mathematical representation of the amplitude and phase of the individual notes that make it up [3].

Similar to Fourier series approximation, the Fourier integral approximation improve as the integration limit increase it is expected that the integral will converges to the real function when the integration limits is increased to infinity.

Physical interpretation: the higher the integration limit means higher frequency sinusoidal component have been included in the approximation. (Similar effect has been observed when larger n is used in Fourier series approximation).

The name MATLAB stands for matrix laboratory. MATLAB was written originally to provide access to matrix software developed by the LINPACK (linear system package) and EISPACK (Eigen system package) projects.

MATLAB is a high-performance language for technical computing it integrates computation, visualization and programming environment. Furthermore MATLAB is a modern programming language environment. It has sophisticated data structure contains built-in editing and debugging tools, and support object oriented programming. These factors make MATLAB an excellent tool for teaching and research. MATLAB has many advantage compared to conventional computer language (e.g.: C, FORTRAN) for solving technical problems. MATLAB is an interactive system whose basic data element is an array that does not require dimensioning. The software package has been commercially available since 1984 and is now considering as a standard tool at most universities and industries worldwide.

It has powerful built-in routines that enable a very wide variety computation it also has easy to use graphic commands that make it visualization of result immediately available. Specific application is collected in package referred to as toolbox. There are toolbox for signal processing symbolic computation, control theory simulation, optimization and several others fields of applied science and engineering.

Bachman et al. [5] a transform on a Fourier transform like integral applied to a function x(t) but with integration only on a finite interval, usually taking to the interval [0,T]. Equivalently it is Fourier transform of a function x(t) multiplied by a rectangular window function that is finite Fourier transform X(w) of a function that is finite interval [0,T] and is given as,

$$X(w) = \frac{1}{\sqrt{2\pi}} \int_0^T X(t) e^{-iwt}\, dt \qquad (1)$$

Grewal in [5], the integral transform of a function g(x) denoted by I [g(x)] is defined as

$$\hat{g}(x) = \int_{x_1}^{x_2} g(x)\, k(s,x) dx \qquad (2)$$

where K(s,x) is called the kernel of the transform and is a known function of s and the function g(x) is called the inverse transform of g^(s).when K(s,x) is replaced by e^{isx} e^{isx}, we have the Fourier transform of f(x) i.e.

$$f(s) = \int_{-\infty}^{\infty} f(x) e^{isx}\, dx \qquad (3)$$

The function is called the inverse Fourier transform of f(s). Some time, we called it an inversion formula corresponding to f(s).

We have finite Fourier sine and cosine transform. These transforms are useful for such a boundary – value problems in which at least two of the boundaries are parallel and separated by a finite distance.
The finite Fourier transform of f(x) in 0<x<c is defined as:

$$Fs(n) = Fs(n) = \int_0^c F(x)\sin\frac{n\pi x}{c}dx \qquad (4)$$

Where n is an integer.
The function f(x) is then called the inverse finite Fourier cosine transform of Fc(n) which is given as

$$F(x) = \frac{1}{c} fc(0) + \frac{2}{c}\sum_{n=1}^{\infty} \quad Fc(n) cos\frac{n\pi x}{c} \qquad (5)$$

2. METHODOLOGY

In this section, we picked some problems from Mathematical Physics text book [4] and Advance Engineering Mathematics by Dass [2], we then solved the problems manually and also computed the result with Matlab software.

3.1. Examples and their Solutions

Example 1. Show That

$$\int_0^\infty \frac{COSUX}{U^2+1} = \frac{\pi}{2}e^{-x} x > 0 \qquad (6)$$

solution

$$F(x) = e^{-x} \quad f(t) = e^{-t} \qquad (7)$$

Substitute F(t) in the formula we have

$$F(x) = \frac{\pi}{2}\int_0^\infty \int_0^\infty f(t) cosutcosuxdtdu \qquad (8)$$

$$F(x) = \frac{\pi}{2}\int_0^\infty du \int_0^\infty f(t) cosutcosuxdt \qquad (9)$$

$$F(x) = \frac{\pi}{2}\int_0^\infty cosuxdu \int_0^\infty e^{-x} cosutdt \qquad (10)$$

$$F(x) = \frac{\pi}{2}\int_0^\infty cosuxdu \left[\frac{e^{-t}}{1+u^2}(-cosut + usinut)\right] \qquad (11)$$

$$e^{-x}\frac{\pi}{2} = \int_0^\infty cosuxdu\left[\frac{1}{1+u^2}\right] \qquad (12)$$

$$e^{-x}\frac{\pi}{2} = \int_0^\infty \frac{cosux}{1+u^2}du \qquad (13)$$

$$\pi e^{-x} = 2\int_0^\infty \frac{cosux}{1+u^2}du \qquad (14)$$

$$\pi\frac{e^{-x}}{2} = \int_0^\infty \frac{cosux}{1+u^2}du \qquad (15)$$

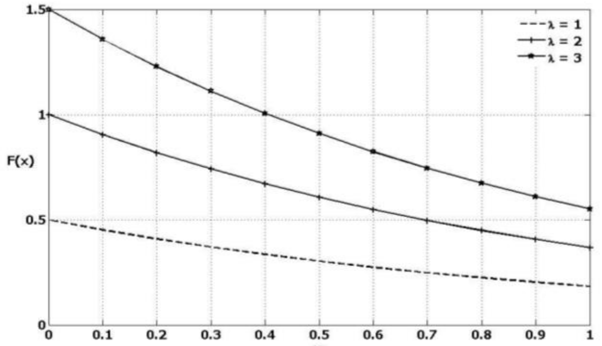

Figure 1: Line Graph

Figure 1 represents the graph of f(x) = $\frac{\pi}{2}e^{-x}$ for different values of π. It is noticed that as π increase f(x) also increase

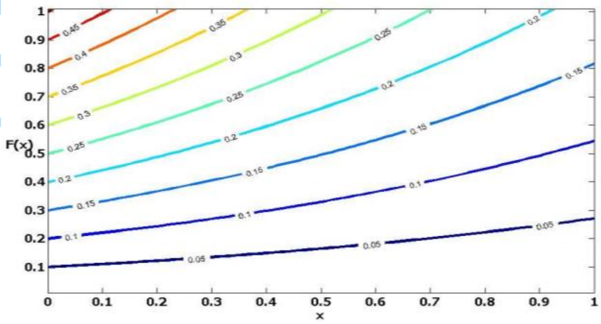

Figure 2: Contour graph

Figure 2 display contour graph of the function f(x)= $\frac{\pi}{2}e^{-x}$ for the range of values of π that is $0 \leq \pi \leq 1$ .from the graph it is clear that the function increases with increasing values of π .

Exmaple 2 Using Fourier Cosine Integral Representation Of An Appropriate Functions Show that:

$$\int_0^\infty \frac{coswx}{K^2 + Fwx}dw = \pi\frac{e^{-kx}}{2k} \qquad (16)$$

SOLUTION

$f(x) = \frac{\pi}{2}\int_0^\infty cosuxdu \int_0^\infty cosutdt$

$f(x) = e^{-kx} f(x) e^{-kt}$ Which implies that

$$f(x) = \frac{\pi}{2}\int_0^\infty [cosux \int_0^\infty e^{-kt}coswtdt\,]\,dx$$

Using integration by part

$$\int e^{-kt}coswtdt = -\frac{e^{-kt}}{k}coswt - \qquad (17)$$

$$-\frac{w}{k}\left[-\frac{e^{kt}}{k}sinwt + \frac{w}{k}\int e^{-kt}coswt + dt\right]$$

$$\int e^{-kt}coswtdt = -\frac{e^{-kt}}{k}coswt - \frac{we^{-kt}}{k^2}sinwt - \frac{w^2}{k^2}\int e^{-kt}coswtdt$$

$$\frac{1+w^2}{k^2}\int e^{-kt}coswtdt = -\frac{e^{-kt}}{k}coswt + \frac{we^{-kt}sinwt}{k^2}$$

$$\int e^{-kt}coswtdt = -\frac{e^{-kt}}{k}coswt + \frac{we^{-kt}sinwt}{k^2}$$

$$\int e^{-kt}coswtdt = -\frac{k^2}{k^2+w^2}\left[\frac{1}{k^2} - ke^{-kt}coswt + we^{-kt}sinwt\right]_0^\infty$$

(18)

$$= \frac{e^{-kt}}{k^2+w^2}[-kcoswt + wsinwt]_0^\infty$$

$$\int_0^\infty e^{-kt}coswt = \left[\frac{e^{-kt}}{k^2+w^2}(-kcoswt + wsinwt)\right]_0^\infty \qquad (19)$$

$$=> 0 - \left[\frac{e^0}{k^2+w^2}(-k\cos 0 + w\sin 0)\right] \qquad (20)$$

$$=> 0 - \left[\frac{1}{k^2+w^2}(-k+0)\right] = \frac{k}{k^2+w^2} \qquad (21)$$

By substitution we have

$$e^{-kx} = \frac{2}{\pi}\int_0^\infty k\frac{coswx}{k^2+w^2}dx \qquad (22)$$

$$\pi e^{-kx} = 2k\int_0^\infty \frac{coswx}{k^2+w^2}dx$$

Divide both sides by $2k$

$$\frac{\pi e^{-kx}}{2k} = \int_0^\infty \frac{coswx}{k^2+w^2}dx. \qquad (23)$$

Hence, the result.

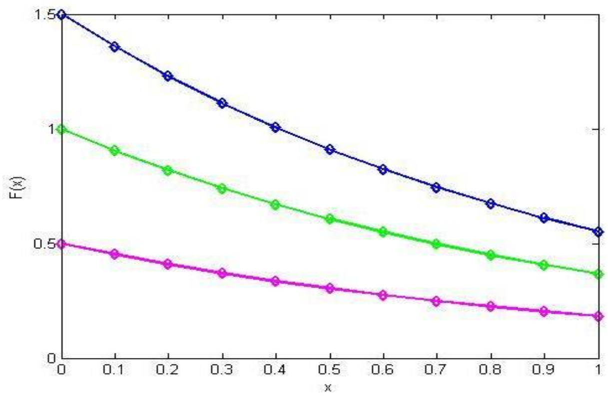

Figure 3: Line Graph

The influence of π on the function $f(x) = \frac{\pi}{2k}e^{-kx}$ is displayed in figure 3. From the figure it is observed that increasing π lead to significant increase in the function.

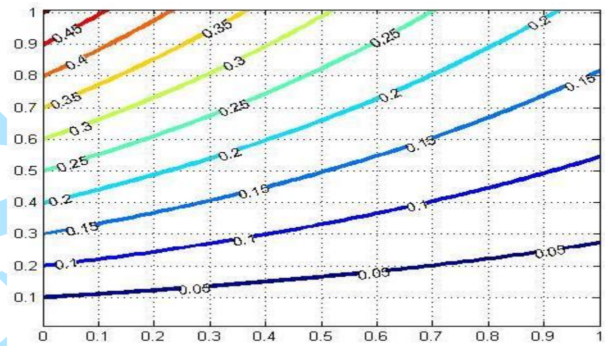

Figure 4: Contour Graph

Figure 4 represent the contour graph of the function $f(x) = \frac{\pi}{2k}e^{-kx}$ for different values π. From the graph it can be seen that small value of π decreases the function while high value of π increases the function.

Example 3. Find the Fourier sine integral for $f(x) = e^{Bx}$ and hence show that

$$\frac{\pi}{2}e^{-Bx} = \int_0^\infty \frac{\pi Sin\pi x}{B^{2+\pi^2}} \qquad (24)$$

Solution

The Fourier Sine Integral For F(X) Is Given by

$$f(x) = \int_0^\infty \int_0^\infty F(t)sin\,\pi t sin\pi x dt d\pi \qquad (25)$$

$$f(x)\,\frac{\pi}{2}\int_0^\infty sin\pi x\int_0^\infty f(x)sin\pi dt\,\}\,d\lambda \qquad (26)$$

$$f(x) = e^{-Bx}\ then\ f(t) = e^{-Bt}$$

Substitute f(x) and f(t) in equation (25) we have

$$F(x) = \frac{\pi}{2}\int_0^\infty \left[\sin\pi x \int_0^\infty e^{-Bt}\sin\lambda t dt\right] d\lambda \tag{27}$$

$$F(x) = \frac{\pi}{2}\int_0^\infty \sin\lambda x d\lambda \left[\frac{e^{-Bx}}{B^{2+\lambda^2}}(-B\sin t - \lambda\cos t)\right]_0^\infty \tag{28}$$

$$F(x) = \frac{\pi}{2}\int_0^\infty \sin\lambda x d\lambda \left[0 + \frac{\lambda}{B^2 + \lambda^2}\right] \tag{29}$$

$$F(x) = \frac{\pi}{2}\int_0^\infty \frac{\lambda\sin\lambda x}{B^2+\lambda^2} d\lambda$$

$$e^{-Bx} = \frac{\pi}{2}\int_0^\infty \frac{\lambda\sin\lambda x}{B^2+\lambda^2} d\lambda \tag{30}$$

$$\pi e^{-Bx} = 2\int_0^\infty \frac{\lambda\sin\lambda x}{B^2+\lambda^2} d\lambda = \pi\frac{e^{-Bx}}{2} = \int_0^\infty \frac{\lambda\sin\lambda x}{B^2+\lambda^2} d\lambda. \tag{31}$$

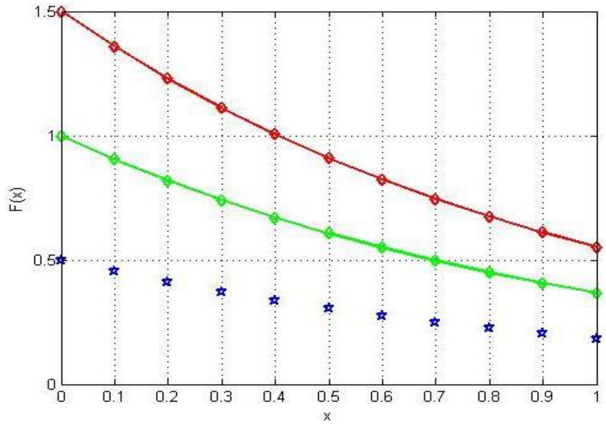

Figure 5: Line Graph

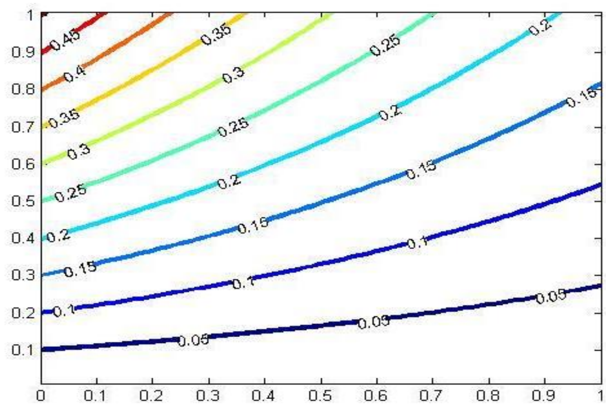

Figure 6: Contour Graph

The influence of π on the function f(x) = $\frac{\pi}{2}e^{-bx}$ is displayed in figure 5 and figure 6 respectively. From these figures it is clear that increasing π lead to significant increase in the function.

Example 4 Express the Function

$$F(x)\begin{cases} 1 \text{ when } |x| \le 1 \\ 0 \text{ when } |x| > 1 \end{cases}$$

Solution

The Fourier integral for f(x) is given as

$$F(x) = \frac{1}{\pi}\int_0^\infty \int_0^\infty \cos\lambda(t-x)dt d\lambda \tag{32}$$

Substitute f(x)=1 in equation 32

$$F(x) = \frac{1}{\pi}\int_0^\infty \int_0^\infty \cos\lambda(t-x)dt d\lambda$$

$$F(x) = \frac{1}{\pi}\int_0^\infty \frac{\sin\lambda(t-x)}{\lambda} /_{-1}^{1} \tag{33}$$

Applying the limit we have

$$F(x) = \frac{1}{\pi}\int_0^\infty \left[\frac{\sin\lambda(1-x)}{\lambda} - \sin\lambda(-(1+x))\right] dx$$

$$F(x) = \frac{1}{\pi}\int_0^\infty \sin\pi - \frac{\lambda x + \sin\lambda(1+\lambda x)}{\lambda} d\lambda \tag{34}$$

$$\sin(\lambda - \lambda x) = \sin\lambda\cos\lambda x - \sin\lambda x\cos\lambda x$$

$$\sin(\lambda + \lambda x) = \sin\lambda\cos\lambda x + \sin\lambda\cos\lambda x$$

$$\sin(\lambda - \lambda x) + \sin(\lambda + \lambda x) = 2\sin\lambda\cos\lambda x$$

Substitute in 34, we have

$$f(x) = \frac{1}{\pi}\int_0^\infty \frac{2\sin\lambda\cos\lambda x}{\lambda} d\lambda$$

$$F(x) = \frac{2}{\pi}\int_0^\infty \frac{\sin\lambda\cos\lambda x}{\lambda} d\lambda$$

$$\frac{\pi F(x)}{2} = \int_0^\infty \frac{\sin\lambda\cos\lambda x}{\lambda} d\lambda$$

$$\frac{\pi}{2}f(x) = \begin{cases} \frac{\pi}{2} \text{ for } x < 1 \\ 0 \text{ for } x > 1 \end{cases}$$

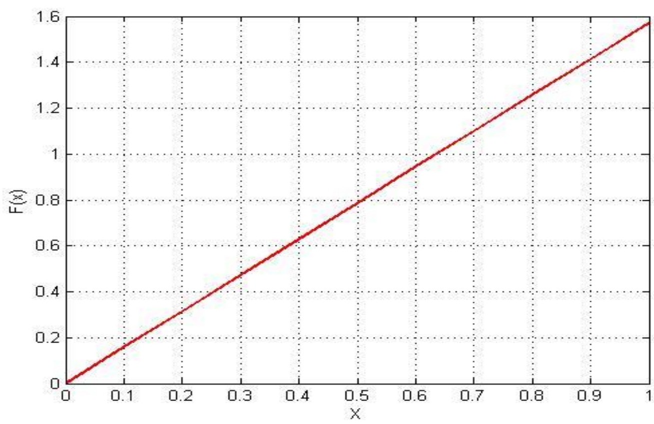

Figure 7: Line Graph

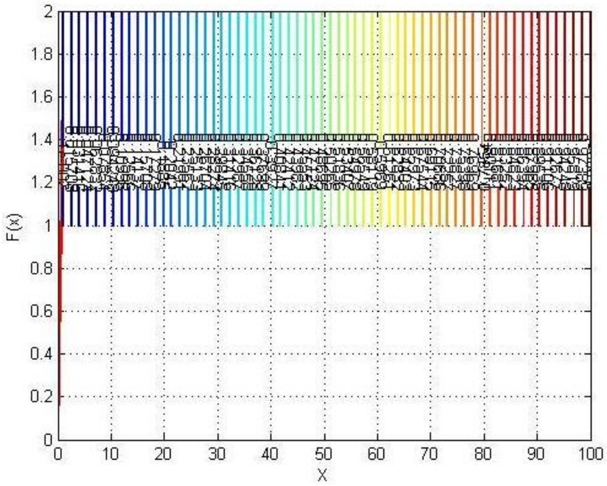

Figure 8: Countour Graph

Figure 7 and figure 8 represent the influence of π on the function $\frac{\pi}{2}$f(x). It is observed that the function increases with increasing value of π (see fig 7 and 8).

5. CONCLUSIONS

So far, we have presented the fundamental theory of Fourier integral and its graphical solution using MATLAB. Conclusively the graphical representation in MATLAB has been founded to be efficient in case where it is applicable. At the point, someone can solve example of Fourier integral with computational/graphical representation of MATLAB based on the algorithms in this project.

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