Numerical experiments on attraction basin

Dr. St. Măruşter, Professor,
"Tibiscus" University of Timişoara

Abstract. The paper contains some numerical experiments concerning the basin of attraction of a tangent parabola method (a particular damping Newton method) for solving nonliniar equations in several variables.
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1 Introduction

In the paper [Mar01] a particular Tangent Parabola Method for solving nonlinear equations in one variable was considered. The basic idea of the method (see also [Sza73], [Ser87]) is to use a tangent parabola (instead of a tangent line) to compute the next iterate. The method has quadratic convergence and generally enlarges the basin of attraction, that is the set of points from which we can start the iterative process. For certain cases the enlargement is significant, as the paper [Cir01] shows for more examples.

For equation in one variable ($f(x) = 0$, $f: I \subseteq R \to R$) the Tangent Parabola Method is given by the following iteration formula

$$x_n = x - \frac{2}{1 + \sqrt{1 + (f(x)/f'(x))^2}} \cdot \frac{f(x)}{f'(x)} \qquad (1)$$

where x is the current iterate and x_n is the next iterate.

This method differs from the Newton method only by the sub-unitary factor $\frac{2}{1+\sqrt{1+(f(x)/f'(x))^2}}$ which multiplies the correction term $f(x)/f'(x)$ of the Newton method. Such a factor has the useful effect of lessening (damping) the undesirable phenomenon of throwing far away the next iterate if $f'(x)$ is small, as it can happen in the Newton method. In fact, the method (1) is a particular *damping Newton method* in which the *damping factor* depend of the current iterate x. Note that this damping factor does not ensure that $|f(x_n)|<|f(x)|$. If the sequence $\{x_k\}$ generated by (1) tends to the solution x^* of the equation $f(x) = 0$ as k tends to infinite, the damping factor tends to the value of one and asymptotic behavior of (1) is like a pure Newton method and quadratic convergence is achieved. Note also that sometime the correction term in Newton method is increased using an positive integer factor, as in *relaxed Newton method* for multiple roots.

For several variables ($F(x) = 0$, $F: D \subseteq R^n \to R^n$) the formula (1) can be written as follows

$$x_n = x - \frac{2F'(x)^{-1}F(x)}{1+\sqrt{1+\left\|F'(x)^{-1}F(x)\right\|^2}} \qquad (1')$$

Note that this is not a genuine generalization of (1). For $n = 2$ ($F(x)=(f_1(x),f_2(x))^T$) such a genuine generalization can be done as follows. Let T_1, T_2 be the points in which a vertical straight line intersects the graphs of the components f_1 and f_2, respectively. Now, draws the elliptic paraboloids with vertical axes tangent to the both surfaces in the points T_1, T_2. The intersection line of these paraboloids is a parabola which intersects the horizontal plane in two points. We can take one of these points as the next iterate.

Two problems arise:

(I) The tangency conditions do not determine the paraboloids and therefore do not determine the intersection parabola. The tangent paraboloids must be chosen in such a way that this parabola intersects the horizontal plane in two real points;

(II) It requires a strategy for select between the two zeros that one which ensures a suitable behavior of resulting sequence.

For one variable case these problems are solved in a very simple way (Mar01). For several variables this way is more difficult and the resulting formula is extremly sophisticated.

In this note we focus on the iterative method (1') which is a formal generalization of (1); in the next we will call them as damping Newton method. Note however that both conditions I and II are satisfied of this formula; indeed, if the jacobian is invertible for all x in D then the next iterate always exists and also it is moved in the same direction on the horizontal plane as the next iterate of Newton method, i.e. we have the required strategy (condition II).

The main purpose of our work is to investigate the question of whether the damping Newton method (1') enlarges the basin of attraction of certain solutions of equations in several variables, as it happens in one variable case. In section 2 we prove the superlinear convergence of the sequence generated by (1'). Section 3 is the main part of the paper; it contains some numerical experiments which answer (partly) to the our question by comparing the damping Newton method and the classical Newton method from the basin of attraction point of view.

We note that the effect of damping Newton-Armijo rule [DS96] about the basins of attraction for the three roots of the complex polynomial z^3-1 and for the solutions of a symmetric two-dimensional nonlinear system was considered in [EG98]. It is showed that the fractal structure of the borders of the basins of attraction is preserved and that fractal dimension is diminished.

2 The superlinear convergence

Consider more general iterations of the form

$$x_n = x - \varpi(x)A(x)^{-1}F(x), \qquad (2)$$

$\varpi : D \subseteq R^n \to R^n$ is a given real function (the damping factor) and A is a mapping from a suitable subset of R^n into $L(R^n)$, that is into the set of linear operators from R^n to R^n; in other words, $A(x)$ is an n x n functional matrix which depends of x. Of course, the iteration (1') is a particular case of (2) in $\varpi(x) = 2/(1+\sqrt{1+\|F'(x)^{-1}F(x)\|^2})$ and $A(x)$=F'(x). We begin with a lemma concerning (2) which is a slight generalization of a similar lemma of [OR70] (lemma 10.2.1 from chapter 10).

Lemma. *Suppose that* $F: D \subseteq R^n \to R^n$ *is F-differentiable at a point* $x^* \in int(D)$ *for which* $F(x^*) = 0$. Let $A: S_0 \to L(R^n)$ *be defined on an open neighborhood* $S_0 \subseteq D$ *of* x^* *and continuous at* x^*, *and assume that* $A(x^*)$ *is nonsingular. Suppose also that the real function* $\varpi : D \subseteq R^n \to R^n$ *is continuous at* x^*. *Then there exists a ball* $S = S(x^*,\delta) \subseteq S_0$, $\delta > 0$, *on which the mapping* $G: S \to R^n$ *given by*

$$G(x) = x - \varpi(x)A(x)^{-1}F(x) \qquad (3)$$

is well-defined; moreover, G is F-differentiable at x^*, *and*

$$G'(x^*) = I - \varpi(x^*)A(x^*)^{-1}F'(x^*)$$

The proof follows almost verbatim the proof of lemma 10.2.1 from [OR70]. We point out only that in our proof must use everywhere $\varpi(x)A(x)^{-1}$ in place of $A(x)^{-1}$ and that it takes place the inequality $|\varpi(x)- \varpi(x^*)| < \varepsilon$.

In particular, if $\varpi(x^*) = 1$ and F is G-differentiable in a neighborhood of x^* and $A(x) = F'(x)$, then $G'(x^*) = 0$ and the sequence given by the generation function (3) converges to x* and has superlinear convergence (see, for example, [OR70], lemma 10.1.6). Now, for the iteration (1'), $\varpi(x) = 2/(1+\sqrt{1+\left\|F'(x)^{-1}F(x)\right\|^2})$ and obviously $\varpi(x^*) = 1$. Also, it can simple see that, if $F'(x^*)$ is invertible, ϖ is a continuous function in x^* and we have

Corollary. *Suppose that* $F: D \subseteq R^n \to R^n$ *has a solution* $x^* \in int(D)$ *and is F-differentiable at* x^* *and that* $F'(x^*)$ *is invertible. Then the sequence* $\{x_k\}$ *given by the iteration* (1') *converges superlinearly to* x^* *for any starting point sufficiently close to* x^*.

3 Numerical experiments

The experiments were made using five nonlinear systems in two variables. Let B denote the basin of attraction of some solution of these systems. Generally, this basin consists in a *basic* or *main* set and a lot of small subsets situated in a certain neighborhood of this main set and which are often spread till infinite (see the figure 3, classical Newton method). If a sequence converges to a fixed point x^* of the generation function G, it always reaches this main domain. In [Mar01] this main basin is defined as follows

Definition 1. *The main attraction domain*, B_m, *for an iterative process given by the generation function* $G: R^n \to R^n$ *is the connected part of* B *which contains the fixed point* x^*.

Remarks. (1) For any local convergent method such a main basin of attraction exists. We note also that if certain conditions on a domain is satisfied and if these conditions ensure the convergence, it says that the method has global convergence on this domain.

(2) The isolated points of B have a slight importance in practice, because the probability to guess such a point (for starting the iteration), is generally low. The same remark can be made for any small

connected parts of B. Therefore, if ε is a positive number, we introduce the following righteous definition of *ε-basin of attraction* (see, also, [Mar01]) :

Definition 2. *Let S_ε be any open ball of radius ε. Then ε-basin of attraction is defined by*

$$B^\varepsilon=\{x: x \in B, \exists S_\varepsilon \text{ such that } x \in S_\varepsilon, S_\varepsilon \subset B\}$$

According to this definition, B^ε selects from B only the connected parts which are sufficiently large; of course, the ε-basin of attraction would be a good measure of an iterative method from the basin of attraction point of view. We can also consider the connected part of B^ε which contain the fixed point x^*. Let B^ε_m denote this part; in what follows we will take it as an acceptable measure.

We consider the following five test nonlinear systems in two dimensions:

System 1:

$$sin(x) + 0.2\, y = 0,$$
$$-sin(y) + x^2 y = 0.$$

System 2:

$$x - sin(x)cosh(y)=0,$$
$$y - cos(x)sinh(x)=0.$$

System 3:

$$x - y^3 = 0,$$
$$y + x^4 - y^5=0.$$

System 4:

$$x + x^3 = 0,$$
$$-y + y^2 - 3\, x^2 y = 0.$$

System 5:

$$x + ln(1+y)^2 - y^3 = 0,$$
$$x^5 - y + x^2 y = 0.$$

The choice of these five systems was rather casual, except the system 2 which was considered in [EG98] and for which the authors studied the effect of Newton-Armijo damping algorithms on the basin of attraction corresponding to the solutions of the system.

Each system has the solution $x^*=(0,0)^T$, i.e the origin of coordinate axes (possible more others solutions), but we estimate only the basins of attraction for this null solution. In our figures, for each system the black regions indicate the approximate basins of attraction in a suitable square regions, both for the classical Newton method (in the left) and for the damped Newton method (1') (in the right). The positions of the solution are indicated by a small white circle. These figures was computed by using a mesh of 180x180 pixels inside in this square; each of these pixels was taken as a starting point in the classical Newton method and in the damped Newton method. The pixel was assigned black color if convergence to the considered solution was identified within 50 iterations. (The jacobians of these systems computed in the point (0,0) are invertible and both classical and damping Newton method have superlinear convergence).

The results of experiments are shown in the figures 1-6, below, corresponding to the systems 1-5. (For the system 2 are draw the basins of attraction in a region of 40x40 (Figure 2) and a part of these basins (Figure 3)). The figures allow to compare the basins of attraction of the classical and damped Newton methods.

Remark. The borders of the basins of attraction seem not to have a fractal structure. In the figure 7 two small pieces of the border of this basin associated with classical Newton method for system 1 are depicted. A square with the side of 0.002, centered in the point (0.626,0.626), was placed to the border of the basin (figure 7(a)). Both the mesh and the iteration number was augmented with a view to increase the accuracy. Figure 7 (b) shows the right extremity of the same basin. No fractal structure was found.

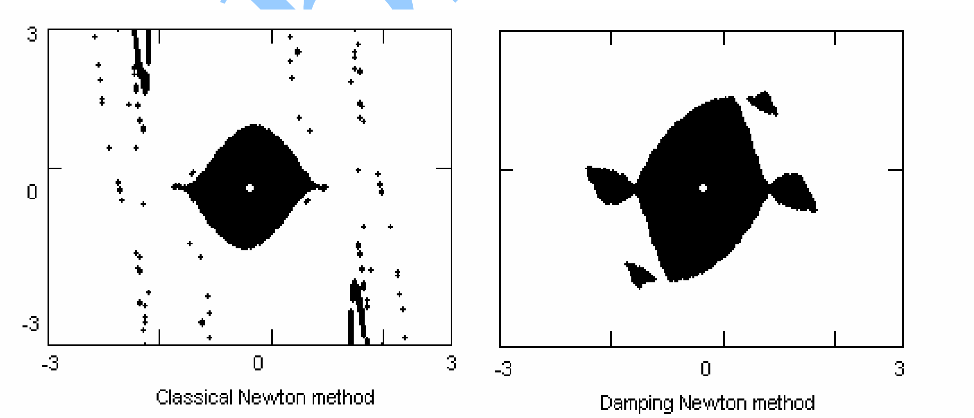

Figure 1. The basin of attraction for system 1.

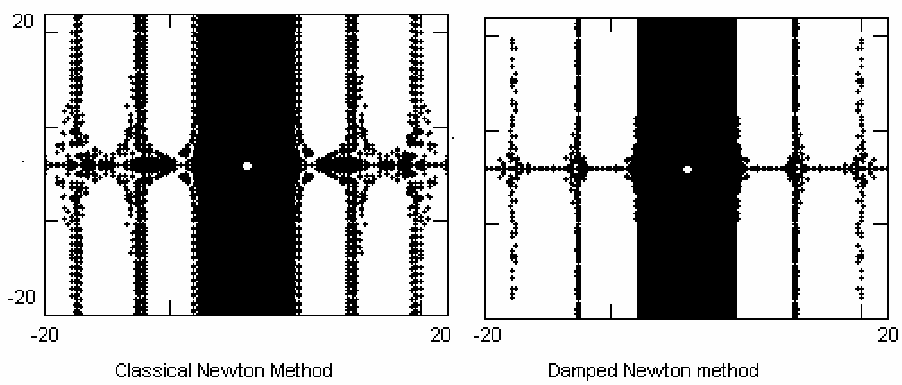

Figure 2. The basin of attraction for system 2.

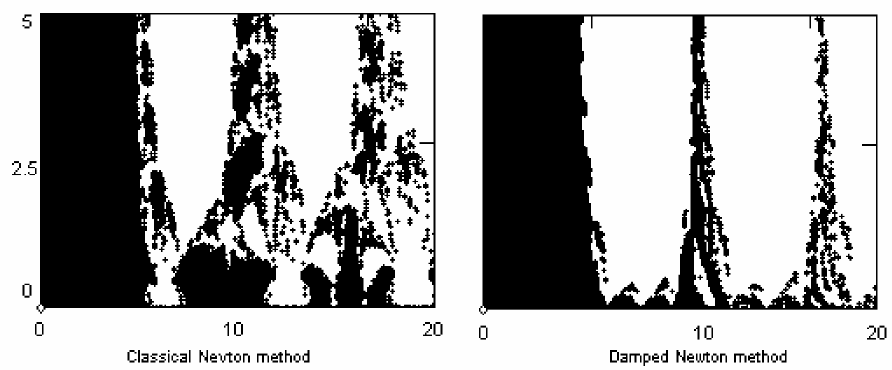

Figure 3. A part of basin for system 2.

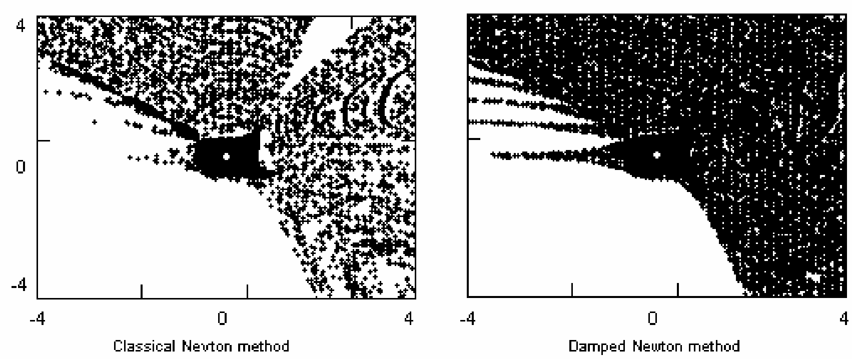

Figure 4. The basin of attraction for system 3.

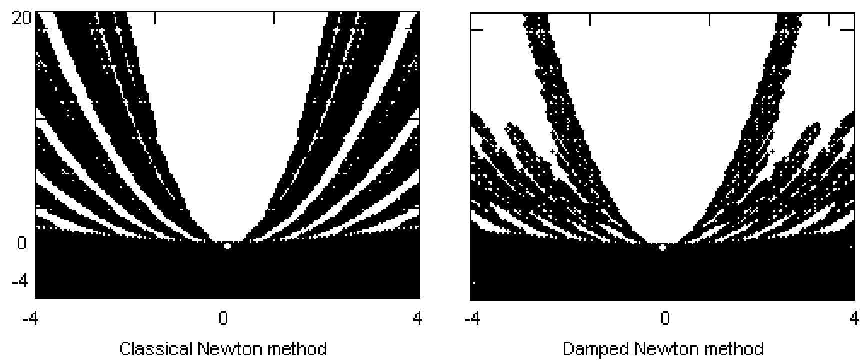

Figure 5. The basin of attraction for system 4.

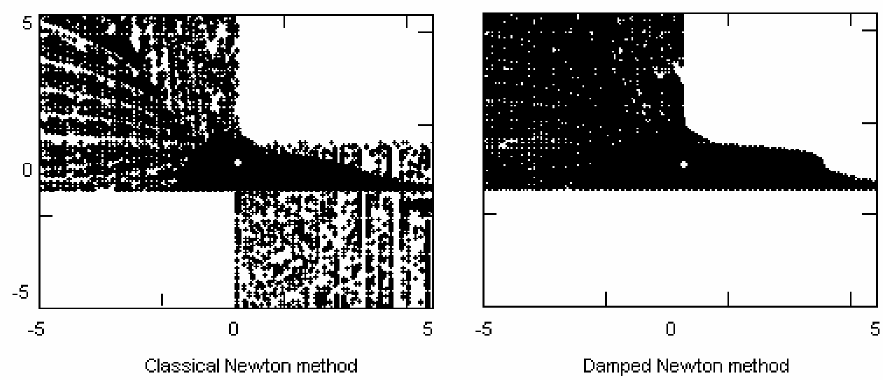

Figure 6. The basin of attraction for system 5.

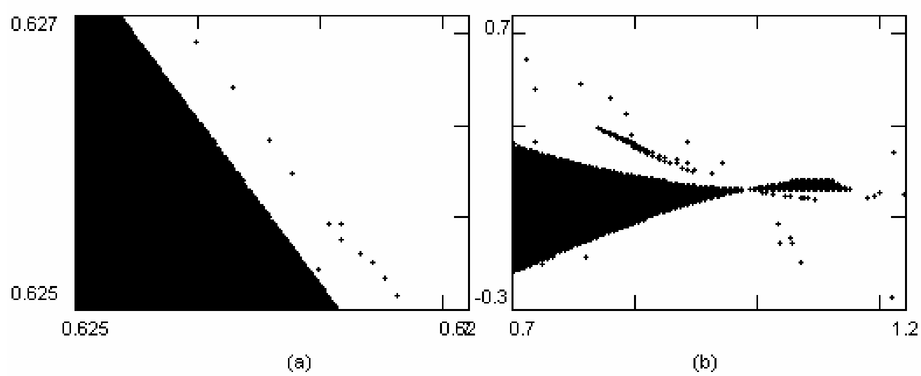

Figure 7. The borders of the basin of system 1.

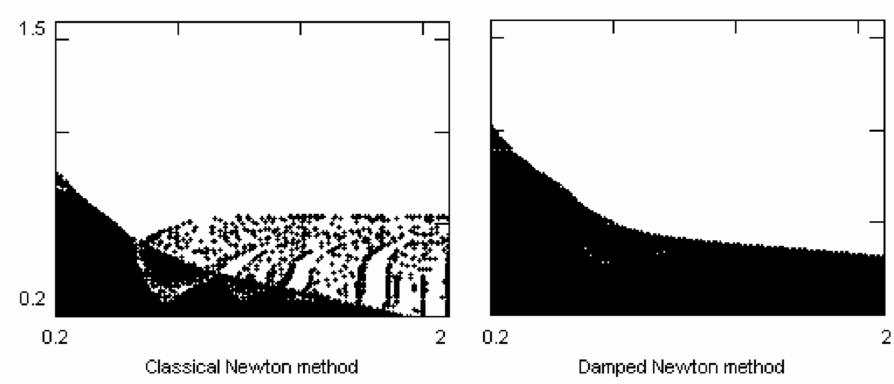

Figure 8. The border of the basin of system 5.

Figures 8(a) and 8(b) show two border regions of the main basin for system 5, classical and damping Newton methods. Also, no fractal structure was found.

4 Conclusions

(1) No significant change of the basin of attraction by applying damping Newton method was encountered. Note that the basin of attraction of the null solution of the system 2 under classical Newton method (figure 2 (a)) is approximately the same as that of the paper [EG98], figure 3.1 (the green region). Also, the damping Newton method (1') and the damping Newton Armijo method have about the same effect; compare figure 3 (b) and figure 3.2 from [EG98].

(2) The main basins of attraction are strong delimited by the rest of basins, both for classical and damping Newton method. This remarkable property can be seen for all considered examples.

(3) The main basins of attraction were enlarged to a certain extent for the majority of examples by applying damping method.

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