

## Image and Transfer Functions

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**ABSTRACT.** We describe three transfer functors  $P, P', P''$  of an inverse exact category which arise from three transfer functions. We concentrate on some of the basic results which emerge from the theory of projections in inverse exact categories.

### 1. Baer\*-categories and exact inverse categories

Inverse categories have been considered by a number of authors: J.Kastl [Kas79], M.Grandis [Gra76], H.J.Hoehnke [Hoe88], M.V.Lawson [Law91] and by the author of this paper ([Sch81]). A category  $C$  is said to be inverse if for each morphism  $f$  there exists a unique morphism  $g$  such that

$$fgf = f \quad \text{and} \quad gfg = g.$$

The morphism  $g$  is called the inverse of  $f$  and we shall denote it by  $f^{-1}$ . The inverse is then nothing but an involution of the inverse category called the canonical involution. An involution of a category is a contravariant endofunctor identical on objects and involutory on morphisms. In a category with an involution  $*$ , if for a morphism  $f$  there exists a morphism  $g$  such that

$$fgf = f, \quad gfg = g, \quad (fg)^* = fg, \quad (gf)^* = gf$$

then it is unique and it is called the Moore-Penrose generalized inverse of  $f$  with respect to  $*$ . The generalized inverse of a morphism  $f$  is denoted by  $f^{(-1)}$ .

An idempotent morphism  $i$  such that  $i^*=i$  is called projection. A category  $B$  with zero object and with involution  $*$  is called a Baer\*-category if for each morphism  $f$  there exists a (unique) projection  $f'$  so that

$$\{g \in B \mid fg = 0\} = f' B.$$

A projection  $i$  of a Baer\*-category is called closed if  $(i')' = i$  (often,  $(f')'$  is denoted by  $f''$ ).

An exact category is a normal and conormal category with kernels and cokernels and with the property that every morphism  $f$  can be written as a composition  $f=pq$  where  $p$  is a monomorphism and  $q$  is an epimorphism.

The main result of this section is the following one:

**Theorem 1.1.** *Let  $C$  be a category in which any morphism has a Moore-Penrose generalized inverse with respect to an involution  $*$ . Then  $C$  is exact if and only if  $C$  is a Baer\*-category with closed projections so that every projection  $i$  can be written as a composition  $i=pq$  where  $p$  is a monomorphism and  $q$  is an epimorphism.*

**Proof.** Assume that  $C$  is exact. Let  $f$  be a morphism of  $C$  and  $u=\ker f$ . Then, the morphism  $i=uu^{(-1)}$  is a projection such that  $fi=0$ . Now, if  $g$  is a morphism of  $C$  such that  $fg=0$ , then there exists a morphism  $h$  such that  $g=uh$  and therefore  $ig=g$ . It follows that  $\{g \in C \mid fg=0\}=iC$ , that is:

$$f' = \ker f (\ker f)^{(-1)}$$

for any morphism  $f$  of  $C$ . Consequently,  $C$  is a Baer\*-category.

Since  $C$  is exact, any projection  $i$  has a mono-epi factorization:  $i=pq$ , and any monomorphism is the kernel of a morphism. It follows that  $p=\ker h$  for some morphism  $h$  of  $C$ . Then  $hi=0$ , and for any morphism  $g$  of  $C$  such that  $hg=0$ , there exists a morphism  $k$  of  $C$  such that  $g=pk$ . Consequently,

$$g = pk = pqpk = i(pk) = ig.$$

This shows that

$$i = h'.$$

Now, it is straightforward to check that for any morphism  $h$  of a Baer\*-category we have  $h' = \{[h']\}'$ . Thus, it follows that any projection  $i$  of a Baer\*-category is closed.

Conversely, if the category  $C$  with Moore-Penrose inverses is a Baer\*-category which closed projections so that every projection can be written as a composition of a monomorphism with an epimorphism, then for any morphism of  $C$ , the mono-epi factorizations

$$f' = p_1 q_1 \quad \text{and} \quad (f^*)' = p_2 q_2$$

implies:

$$p_1 = \ker f \quad \text{and} \quad q_2 = \text{co ker } f.$$

Thus,  $C$  is a category with kernels and cokernels. To show that  $C$  is normal and conormal, let  $u$  be a monomorphism of  $C$  and let  $v$  be an epimorphism of  $C$ . It is straightforward to check that

$$u = \ker(u^*)' \quad \text{and} \quad v = \text{co ker } v'.$$

So,  $C$  is normal (every monomorphism is the kernel of some morphism) and conormal (every epimorphism is the cokernel of some morphism).

Now, let  $f$  be a morphism of  $C$  and let

$$[(f^*)]' = pq$$

be a mono-epi factorization of the projection  $[(f^*)]'$ . Then the monomorphism  $p$  is an epimorphic image of  $f$ .

Consequently,  $C$  is an exact category.

The canonical involution  $*$  of an inverse category is defined by:  $f^* = f^{-1}$ . It is easy to see that  $f^{-1}$  is the Moore-Penrose inverse of  $f$  with respect to the canonical involution of an inverse category. So, an inverse category is exact if and only if it is a Baer\*-category with closed projections in which every projection has a mono-epi factorization.

The category of partial bijections between sets is an exact inverse category. Another exact inverse category is the following one: the objects are all the sets with base points; the morphisms from  $(A,a)$  to  $(B,b)$  are the

set functions  $f$  from  $A$  to  $B$  such that for any  $y \in B$  there exists  $x \in A$  so that  $f(x)=y$  and  $f(a)=b$ ; the composition is the usual composition of maps. An inverse monoid  $S$  with zero adjoined is an inverse category with two objects. This inverse category is exact if and only if  $S$  is a group.

## 2. The image functor $P$

Transfer functions were introduced by Grandis [Gra77]. The orthodox expansion of a regular category with involution is constructed based on transfer functions. If  $R$  is a regular category (i.e. for any morphism  $f$  there is a morphism  $g$  such that  $fgf=f$ ) with an involution  $*$ , and  $f$  is a morphism of  $R$  from  $A$  to  $B$  then the function  $T(f) : Hom_R(A, A) \rightarrow Hom_R(B, B)$  defined by

$$T(f)(h) = fhf^* \quad \text{for any } h \in Hom_R(A, A)$$

is called the transfer function for  $f$ .

Now, let  $C$  be an exact inverse category and let  $*$  be the canonical involution on  $C$ . If  $f$  is a morphism of  $C$  from  $A$  to  $B$  we call the restriction on  $P(A)$  (the set of all projections from  $A$  to  $A$ ) of the transfer function  $T(f)$ , the image function for  $f$ . The functor  $P:C \rightarrow \text{Ens}$  defined by:

$$A \in \text{Ob}C \longrightarrow P(A) \quad ; \quad f \in Hom_C(A, B) \longrightarrow P(f) \in Hom_{\text{Ens}}(P(A), P(B)),$$

where  $P(f):P(A) \rightarrow P(B)$  is the image function for  $f$ :

$$P(f)(i) = fif^* \quad (i \in P(A))$$

is a covariant functor.

**Theorem 2.1** *If  $u \in Hom_C(X, A)$  is a monomorphism,  $f \in Hom_C(A, B)$  is a morphism and*

$$P(f)(uu^*) = pp^*$$

*with  $p:I \rightarrow B$  monomorphism, then  $p$  is the image of  $fu$  (i.g.  $f(X)=I$ ).*

**Proof.** We have:

$$fu = fu(fu)^* fu = fuu^* f^* fu = P(f)(uu^*) fu = pp^* fu = pq$$

where  $q=pp^*fu$ . Now, let  $fu=st$  a factorization of  $fu$  with  $s$  monomorphism. Then,

$$ss^* pp^* fu = ss^* fu = ss^* st = st$$

and therefore,

$$t = s^* pp^* fu.$$

It follows:

$$ss^* pp^* = ss^* P(f)(uu^*) = ss^* fuu^* f^* = ss^* pp^* fuu^* f^* = stu^* f^* = fuu^* f^* = pp^*$$

Consequently:

$$ss^* p = p.$$

Thus,  $(I,p)$  is the smallest subobject of  $B$  which  $fu$  factors through. So,  $p$  is the image of  $fu$ .  $\square$

Taking into account Theorem 2.1 we say that  $P$  is the image functor of the exact inverse category  $C$ . Some of the elementary properties of the image functor appear in the next theorem.

**Theorem 2.2** *Let  $C$  be an exact inverse category and  $P$  the image functor of  $C$ . The following properties are true:*

- (i)  $P$  preserve monomorphisms and epimorphisms;
- (ii)  $P(f)(0)=0$  and  $P(f)(1)=ff^*$  for any morphism  $f$  of  $C$ ;
- (iii)  $P(f)(f^*f)=ff^*$  for any morphism  $f$  of  $C$ ;

**Proof.** (i) Let  $f$  be a monomorphism of  $C$  from  $A$  to  $B$ . If  $P(f)(i) = P(f)(j)$  (where  $i, j \in P(A)$ ), that is  $fi^* = fj^*$ , then  $if^* = jf^*$  since  $f$  is monomorphism. This result implies

$$(fi)^* = (jf)^* \Rightarrow fi = fj \Rightarrow i = j$$

and so,  $P(f)$  is a monomorphism of  $\text{Ens}$ .

Now, let  $f$  be an epimorphism of  $C$  from  $A$  to  $B$  and  $j \in P(B)$ . Since  $ff^* = 1_B$  it follows

$$(f^*jf)(f^*jf) = (f^*jf) \quad \text{and} \quad (f^*jf)^* = f^*jf.$$

Hence  $f^*jf \in P(A)$  and so

$$P(f)(f^*jf) = ff^*jff^* = j.$$

Thus  $P(f)$  is an epimorphism of  $\text{Ens}$ .

(ii) and (iii) are obviously.  $\square$

An inverse semigroups in which every element is an idempotent are precisely the meet semilattices. If  $S$  is an inverse semigroup in which every element is an idempotent then a relation  $\leq$  on  $S$  defined by

$$e \leq f \Leftrightarrow e = ef (= fe)$$

is a partial order on  $S$  and  $e \cap f = ef$ . Conversely, let  $(S, \leq)$  be a meet semilattice. It is routine to check that  $S$  is a commutative semigroup with respect to the operation  $\cap$ , the greatest lower bound. Clearly,  $e = e \cap e$  for each element  $e \in S$ . Thus  $(S, \cap)$  is an inverse semigroup in which every element is idempotent.

In an exact inverse category  $C$  for any object  $A$ ,  $P(A)$  is an inverse semigroup in which every element is an idempotent. Thus  $P(A)$  is a meet semilattice. We have:

**Theorem 2.3.** *Let  $C$  be an exact inverse category and let  $f$  be a morphism of  $C$  from  $A$  to  $B$ . Then:*

- (i)  $P(f)$  is an inverse semigroups homomorphism;

- (ii)  $P(f)$  has the order preserving property;
- (iii)  $P(f)(i) \leq ff^*$  for any  $i \in P(A)$
- (iv)  $P(f)(i) = ff^*$  if  $i \geq f^*f$

**Proof.** (i).

$$P(f)(ij) = f i j f^* = f i f^* f j f = P(f)(i) \cdot P(f)(j) \quad \text{for any } i, j \in P(A).$$

(ii).

$$\begin{aligned} i \leq j &\Rightarrow ij = i \Rightarrow P(f)(i) \cdot P(f)(j) = f i f^* f j f = f i j f^* = f i f^* = P(f)(i) \Rightarrow \\ &\Rightarrow P(f)(i) \leq P(f)(j). \end{aligned}$$

(iii).

$$P(f)(i) \cdot ff^* = f i f^* ff^* = f i f^* = P(f)(i) \Rightarrow P(f)(i) \leq ff^*.$$

(iv).

$$\begin{aligned} i \geq f^*f &\Rightarrow i f^* f = f^* f \Rightarrow P(f)(i) \cdot ff^* = f i f^* ff^* = ff^* ff^* = ff^* \Rightarrow \\ &\Rightarrow ff^* \leq P(f)(i). \end{aligned}$$

By (iii) it is now clear that if  $i \geq f^*f$  then  $P(f)(i) = ff^*$ . □

### 3. The inverse image functor $P'$

In the case of exact inverse categories, we have another “transfer functions”. Let  $C$  be an exact inverse category and let  $f$  be a morphism of  $C$  from  $A$  to  $B$ . We call the function  $P'(f): P(B) \rightarrow P(A)$  defined by

$$P'(f)(j) = (j' f)' \quad (j \in P(B))$$

the inverse image function for  $f$ . The functor  $P': C \rightarrow Ens$  defined by

$$A \in Ob C \longrightarrow P(A) \quad ; \quad f \in Hom_C(A, B) \longrightarrow P'(f) \in Hom_{Ens}(P(B), P(A))$$

is a contravariant functor.

**Theorem 3.1.** *If  $v \in \text{Hom}_C(A, B)$  is a monomorphism,  $f \in \text{Hom}_C(A, B)$  is a morphism and*

$$P'(f)(vv^*) = pp^*$$

*with  $p : \cdot \rightarrow A$  monomorphism, then the diagram*

$$\begin{array}{ccc} \cdot & \xrightarrow{v^*fp} & \cdot \\ \downarrow p & & \downarrow v \\ A & \xrightarrow{f} & B \end{array}$$

*is a pullback.*

**Corollary 3.2.** *Let*

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \uparrow u & & \uparrow v \\ X & & Y \end{array}$$

*be a diagram of  $C$  with  $u$  and  $v$  monomorphisms. Then*

- (i)  $f(X) = Y$  *if and only if*  $P(f)(uu^*) = vv^*$
- (ii)  $f^{-1}(Y) = X$  *if and only if*  $P'(f)(vv^*) = uu^*$

*where  $f(X)$  denotes the image of the composition  $X \xrightarrow{u} A \xrightarrow{f} B$  and  $f^{-1}(Y)$  is the inverse image of  $Y$  in the sense of Mitchell [Mit65].*

The basic structural properties of  $P'$  are similar with the properties (see Theorems 2.2. and 2.3.) of  $P$  :

**Theorem 3.3.** *Let  $C$  be an exact inverse category and  $P'$  the inverse image functor of  $C$ . The following properties are true:*

- (i)  $P'(f)$  *is a monomorphism (epimorphism) if and only if  $f$  is an epimorphism (monomorphism)*
- (ii)  $P'(f)(0) = f'$  *and*  $P'(f)(1) = 1$  *for any morphism  $f$  of  $C$ ;*
- (iii)  $P'(f)(ff^*) = 1$  *for any morphism  $f$  of  $C$ ;*



**Theorem 3.4.** *Let  $C$  be an exact inverse category and let  $f$  be a morphism of  $C$  from  $A$  to  $B$ . Then:*

- (i)  $P'(f)$  is an inverse semigroups homomorphism;
- (ii)  $P'(f)$  has the order preserving property;
- (iii)  $P'(f)(j) \geq f'$  for any  $j \in P(B)$
- (iv)  $P'(f)(j) = 1$  if  $j \geq ff^*$

The following properties are connection properties between  $P$  and  $P'$ :

**Theorem 3.5.** *let  $C$  be an exact inverse category. Then*

- (i)  $P'(f) = P(f^*)$  if and only if  $f$  is a monomorphism;
- (ii)  $P(f) = P'(f^*)$  if and only if  $f$  is an epimorphism;
- (iii)  $P(f)P'(f)P(f) = P(f)$  and  $P'(f)P(f)P'(f) = P'(f)$  for any morphism  $f$  of  $C$ .

#### 4. The transfer functor $P''$

We now turn to another transfer function in the development of a new transfer functor  $P''$ . If  $C$  is an inverse exact category, then the functor  $P'': C \rightarrow \text{Ens}$  defined by

$$A \in \text{Ob}C \longrightarrow P(A) \quad ; \quad f \in \text{Hom}_C(A, B) \longrightarrow P''(f) \in \text{Hom}_{\text{Ens}}(P(B), P(A))$$

where

$$P''(f)(j) = (jf)''$$

is a contravariant functor.

Some of the elementary properties of  $P''$  appear in the next theorems.

**Theorem 4.1.** *Let  $C$  be an exact inverse category and  $P''$  the transfer functor defined above. The following properties are true:*

- (i)  $P''(f)$  is a monomorphism (epimorphism) if and only if  $f$  is an epimorphism (monomorphism)

- (ii)  $P''(f)(0) = 0$  and  $P''(f)(1) = f''$  for any morphism  $f$  of  $C$ ;
- (iii)  $P''(f)((f^*)') = 0$  for any morphism  $f$  of  $C$ ;

**Theorem 4.2.** *Let  $C$  be an exact inverse category and let  $f$  be a morphism of  $C$  from  $A$  to  $B$ . Then:*

- (v)  $P''(f)$  is an inverse semigroups homomorphism;
- (vi)  $P''(f)$  has the order preserving property;
- (vii)  $P''(f)(j) \leq f''$  for any  $j \in P(B)$
- (viii)  $P''(f)(j) = 0$  if  $j \leq (f^*)'$

We note that between the transfer functions defined above they are interesting connections. For example, it is easy to establish that for any morphism  $f : A \rightarrow B$  of the inverse exact category  $C$ , we have:

$$P''(f)(j) = (P'(f)(j'))' \quad (\forall j \in P(B)).$$

This connection give rise equivalences between Theorems of Sections 3 and 4 (for example: Theorems 3.3 and 4.1., or 3.4. and 4.2.). We prove here some such equivalences:

(i)  $P'(f)$  is a monomorphism (epimorphism)  $\Leftrightarrow P''(f)$  is a monomorphism (epimorphism)

( $\Rightarrow$ ) If  $P'(f)$  is a monomorphism then:

$$\begin{aligned} P''(f)(j_1) = P''(f)(j_2) &\Rightarrow (P'(f)(j_1'))' = (P'(f)(j_2'))' \Rightarrow P'(f)(j_1') = P'(f)(j_2') \Rightarrow \\ &\Rightarrow j_1' = j_2' \Rightarrow j_1 = j_2. \end{aligned}$$

( $\Leftarrow$ ) If  $P''(f)$  is a monomorphism then:

$$\begin{aligned} P'(f)(j_1) = P'(f)(j_2) &\Rightarrow (P'(f)(j_1))' = (P'(f)(j_2))' \Rightarrow P''(j_1') = P''(j_2') \Rightarrow \\ &\Rightarrow j_1' = j_2' \Rightarrow j_1 = j_2. \end{aligned}$$

(ii)  $P'(f)(1) = 1 \Leftrightarrow P''(f)(0) = 0$ ;

( $\Rightarrow$ )  $P'(f)(1) = 1$  implies:

$$P''(f)(0) = (P'(f)(0'))' = (P'(f)(1))' = 1' = 0.$$

( $\Leftarrow$ )  $P''(f)(0) = 0$  implies:

$$(P'(f)(1))' = (P'(f)(0'))' = P''(f)(0) = 0 \Rightarrow P'(f)(1) = 1.$$

(iii)  $P'(f)(0) = f' \Leftrightarrow P''(f)(1) = f''$ .

( $\Rightarrow$ )  $P'(f)(0) = f'$  implies:

$$P''(f)(1) = (P'(f)(1'))' = (P'(f)(0))' = f''.$$

( $\Leftarrow$ )  $P''(f)(1) = f''$  implies:

$$(P'(f)(0))' = (P'(f)(1'))' = P''(f)(1) = f'' \Rightarrow P'(f)(0) = f'. \quad \square$$

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