The Influence of the Resistance at the Advancing Motion of the Material Point Subjected to the Gravitational Force

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ABSTRACT. The paper presents differential equations of motion particle in stationary gravitational field with considerations air resistance. In the numerical solution of the equations of motion are considerate $H = 200 \cdot 10^3 m$, $k_0 = 10^{-9}$ and are obtain revolution period (Pr) and radial distance r(t).
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1. Differential equations of the motion of the material point

In the polar coordinate system with the versors $\bar{i}_r$, $\bar{i}_\theta$, (fig.1), two forces operate:

- The universal attraction force (the central force), $\overline{F}_c$;
- The resistance force at the advancing motion, $\overline{F}_r$.

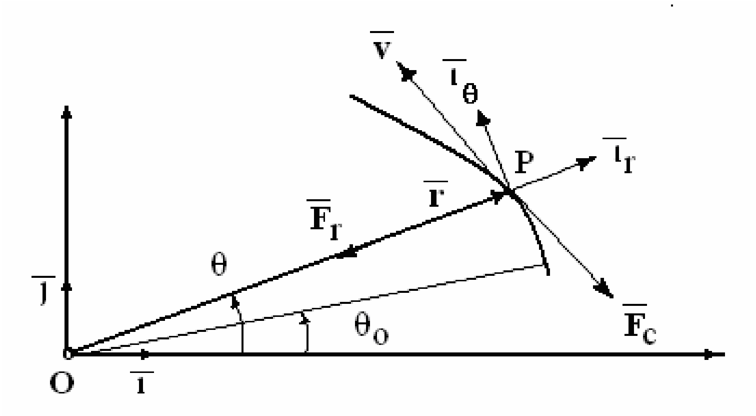

Fig.1. Forces acting upon the material point P

The $\bar{i}_r$ and $\bar{i}_\theta$ versors are written as:

$$\bar{i}_r = \bar{i} \cdot \cos(\theta) + \bar{j} \cdot \sin(\theta), \quad (1)$$

$$\bar{i}_\theta = -\bar{i} \cdot \sin(\theta) + \bar{j} \cdot \cos(\theta), \quad (2)$$

while the derivates with respect to time are:

$$\frac{d\bar{i}_r}{dt} = \dot{\theta} \cdot \bar{i}_\theta, \quad \frac{d\bar{i}_\theta}{dt} = -\dot{\theta} \cdot \bar{i}_r \quad . \quad (3)$$

The speed and the acceleration of the material point P is expressed as:

$$\bar{v}(P,t) = \frac{d\bar{r}}{dt} = \dot{r} \cdot \bar{i}_r + r \cdot \dot{\theta} \cdot \bar{i}_\theta, \quad (4)$$

$$\bar{a}(P,t) = \frac{d\bar{v}}{dt} = (\ddot{r} - r \cdot \dot{\theta}^2) \cdot \bar{i}_r + (2 \cdot \dot{r} \cdot \dot{\theta} + r \cdot \ddot{\theta}) \cdot \bar{i}_\theta . \quad (5)$$

From the fundamental equation of the material point dynamics:

$$m \cdot \bar{a} = \bar{F}_c + \bar{F}_r, \tag{6}$$

Considering the expressions:

$$\bar{F}_c = -F \cdot \frac{\bar{r}}{r} = -\frac{\gamma \cdot M \cdot m}{r^2} \cdot \frac{\bar{r}}{r}, \tag{7}$$

where $\gamma \cdot M = g_0 \cdot R^2$, $g_0 = 9{,}81 \left(\frac{m}{s^2}\right)$, $R=6370 \cdot 10^3$ (m),

$$\bar{F}_r = -k \cdot m \cdot \bar{v}, \tag{8}$$

the system of differential equations results:

$$\ddot{r} = r \cdot \dot{\theta}^2 - \frac{g_0 \cdot R^2}{r^2} - k \cdot \dot{r}, \tag{9}$$

$$\ddot{\theta} = -\frac{\dot{\theta}}{r} \cdot (2 \cdot \dot{r} + k \cdot r), \tag{10}$$

To which the initial conditions are attached:

$$r(0) = r_0, \quad \dot{r}(0) = \dot{r}_0, \tag{11}$$

$$\theta(0) = \theta_0, \quad \dot{\theta}(0)=\dot{\theta}_0. \tag{12}$$

2. The numeric solution of the system of differential equations (9), (10) under the hypothesis that $k = k_0 = \text{const}$.

From the differential equation (10) with the initial conditions (11), (12), through integration, it results:

$$\dot{\theta} = \frac{\dot{\theta}_0 \cdot r_0^{\ 2}}{r^2} \cdot \exp(-k_0 \cdot t), \tag{13}$$

which introduced in (9) obtains:

$$\ddot{r} = \frac{\dot{\theta}_0 \cdot r_0^{\ 4}}{r^r} \cdot \exp\left(-2 \cdot k_0 \cdot t\right) - k_0 \cdot \dot{r} - \frac{g_0 \cdot R^2}{r^2}. \tag{14}$$

From the differential equations (13) and (14) a system of three differential equations of first degree is deduced:

$$\frac{d\theta(t)}{dt} = \frac{\dot{\theta}_0 \cdot r_0^{\ 2}}{r(t)^2} \cdot \exp(-k_0 \cdot t), \quad \theta(0)=\theta_0, \tag{15}$$

$$\frac{dr(t)}{dt} = w(t), \quad r(0) = r_0, \tag{16}$$

$$\frac{dw(t)}{dt} = \frac{\dot{\theta}_0 \cdot r_0^{\ 4}}{r(t)^3} \cdot \exp\left(-2 \cdot k_0 \cdot t\right) - k_0 \cdot \dot{r}(t) - \frac{g_0 \cdot R^2}{r(t)^2}, \quad w(0) = \dot{r}(0) = \dot{r}_0 . \tag{17}$$

<u>Case study 1</u>

The following data is considered:

$$R = 6370 \cdot 10^3 \quad (m), \quad H = 200 \cdot 10^3 \quad (m),$$

$$r_0 = R + H, \quad r_0 = 6{,}57 \cdot 10^6,$$

$$k_0 = 10^{-9}, \; g_0 = 9{,}81 \; \left(\frac{m}{s^2}\right), \; \gamma \cdot M = g_0 \cdot R^2,$$

$$\dot{\theta}_0 = \theta_0 p = \frac{1}{r_0} \cdot \sqrt{\frac{\gamma \cdot M}{r_0}}, \quad \theta_0 p = 1{,}185 \cdot 10^{-3}, \quad C = \theta_0 p \cdot r_0^{\ 2}.$$

For the numerical solution of the differential equations system (15), (16), (17) the following Mathcad algorithm is used:

$$T = 5300$$

Given

$$\frac{d}{dt}\theta(t) = \frac{C}{r(t)^2}\cdot \exp(-k_0 \cdot t) \quad \theta(0) = 0,$$

$$\frac{d}{dt}r(t) = w(t) \quad r(0) = r_0,$$

$$\frac{d}{dt}w(t) = \frac{C^2}{r(t)^3}\cdot \exp\left(-2\cdot k_0 \cdot t\right) - k_0 \cdot w(t) - \frac{g_0 \cdot R^2}{r(t)^2} \quad w(0) = 0$$

$$\begin{bmatrix} f \\ g \\ h \end{bmatrix} = \text{Odesolve}\left[\begin{bmatrix} \theta \\ r \\ w \end{bmatrix}, t, T\right].$$

The $\theta(t)$, $r(t)$ solutions allow to determine the revolution period:

$$\Pr = 5300,66441 \ \text{(s)}, \quad f(5300,66441) = 6,28000001$$

and the variation in time of the distance r(t), to the attractive centre M, represented in fig.2:

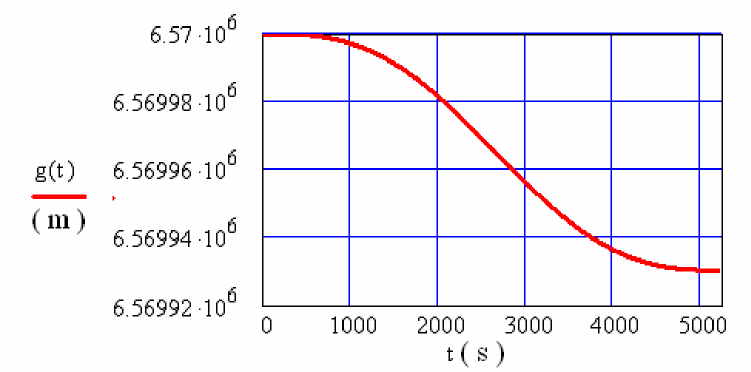

Fig.2. The variation of the distance r(t) during the revolution period in case $k = k_0 = const.$

3. The numerical solution of the differential numeric system (9), (10) under the hypothesis $k = k_0 \cdot (1 + 1/r)$

We introduce in the differential equation (9), (10) $k = k_0 \cdot (1 + 1/r)$ and replace the system of the two differential equations of second degree with the system of four differential equations of first degree:

$$\frac{d\theta(t)}{dt} = T_1(t), \quad \theta(0)=\theta_0, \tag{18}$$

$$\frac{dr(t)}{dt} = R_1(t), \quad r(0) = r_0, \tag{19}$$

$$\frac{dT_1(t)}{dt} = \frac{-T_1(t)}{r(t)} \cdot \left(2 \cdot R_1(t) + k_0 \cdot r(t) + k_0\right), \quad T_1(0) = \dot{\theta}(0) = \dot{\theta}_0, \tag{20}$$

$$\frac{dR_1(t)}{dt} = r(t) \cdot T_1(t)^2 - k_0 \cdot \left(1 + \frac{1}{r(t)}\right) \cdot R_1(t) - \frac{g_0 \cdot R^2}{r(t)^2}, \quad R_1(0) = \dot{r}(0) = \dot{r}_0 \quad . \tag{21}$$

Conclusions

We consider the data in the case study 1: for the numeric solution of the differential equations system (18), (19), (20), (21).

Using the Mathcad function Odesolve:

$$\begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{pmatrix} = \text{Odesolve}\left[\begin{pmatrix} \theta \\ r \\ T_1 \\ R_1 \end{pmatrix}, t, T\right]$$

the revolution period is determined

$$\text{Pr} = 5300{,}66441 \ \ (\text{s}), \ \ \text{f1}(5300{,}66441) = 6{,}28000001$$

and the variation in time of the distance r(t), to the attractive center M, represented in fig.3:

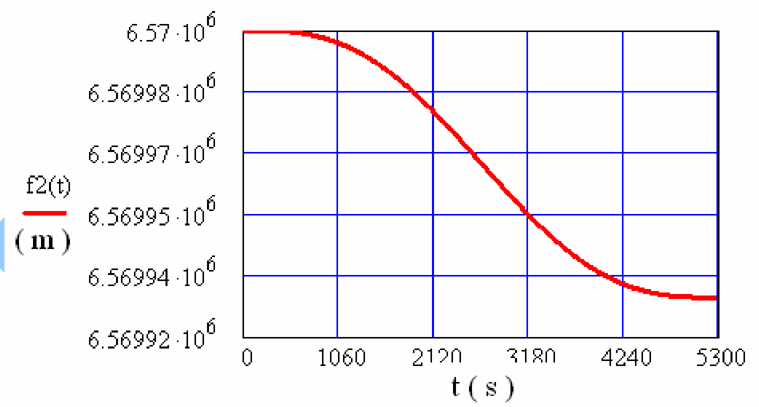

Fig.2. The variation of the distance r(t) during the revolution period in the case $k = k_0 \cdot (1 + 1/r)$

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