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Constructing and Exploring Triangles with GeoGebra

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ABSTRACT: We look at a new approach to mathematical constructions. Using GeoGebra, we first do dynamic analysis and exploration. Again with GeoGebra, we then do a strict compass and straightedge constructions. Finally and yet again with GeoGebra, we do dynamic testing and discussion of proof. In this paper, we work through a complete example of construction a triangle and discuss our results of using this method in the classroom.
KEYWORDS: construction, compass, straightedge, triangle, GeoGebra, median

1 Goal

Our goal here is to illustrate how one can use the free dynamic mathematics software GeoGebra [Geo00] in the classroom to further understanding about advanced compass and straightedge constructions. This continues our research in using GeoGebra both as an exploratory tool and as a tool to do rigorous and precise mathematics [FS10]. With this method, we (a) encourage the student to explore all aspects of a construction, (b) understand the requirements of the construction itself, (c) use a full range of tools to test possibilities for constructing, (d) perform the construction using only GeoGebra compass and straightedge tools [FS09], (e) test the stability of the construction and (f) analyze the conditions for existence and uniqueness of the construction. In this paper, we work through a complete example of the construction of a triangle (from its medians) and compare the understanding and interest of our students between using this and the classical method in the actual classroom.

2 Sample Task

Construct a triangle given its three medians t_a, t_b and t_c.
Base your construction on the definition of the median and the fact that the three medians of a triangle meet at a single point T and T is 2/3 of the way along each of the medians. (T is called the centroid.)

2.1 Initial Analysis and Analysis of Necessary Auxiliary Tasks

As always, the first step is thinking - that is, without GeoGebra. We need to construct a triangle. We have three medians t_a, t_b and t_c. We need to know the definition that a median is a line segment from vertex to the opposite midpoint. At this stage, we make a sketch of the problem.

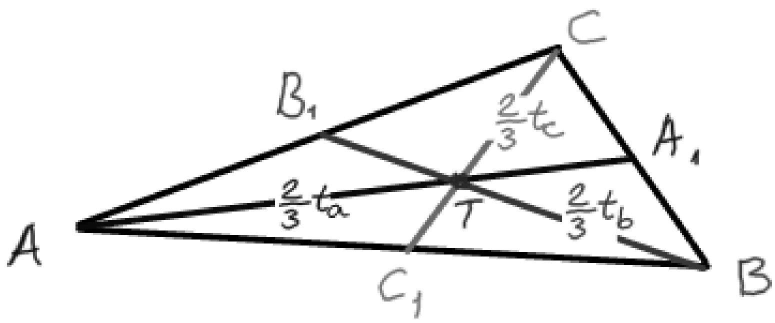

Figure 1. Sketch of Problem

It is clear that we will also need to know the 1/3 lengths of the medians. So – in addition to basic construction techniques – for this construction technically we do need to know the auxiliary compass and straightedge construction "trisect a line segment", which is not discussed here.

2.2 Starting the Construction with GeoGebra

Now the question is – what can we draw with "certainty"?

- We can draw 3 lengths t_a, t_b and t_c (see Figure 2).

On our GeoGebra worksheet, we draw and divide 3 segments into thirds. (Make sure the 3 line segments satisfy the triangle inequality (see Section 4))

By drawing these lengths in GeoGebra like this, (a) we can use *Compass tool* "properly" and (b) we can test our construction dynamically.

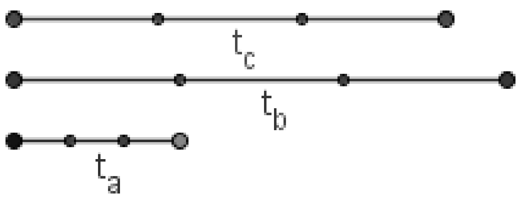

Figure 2: The median lengths and their thirds

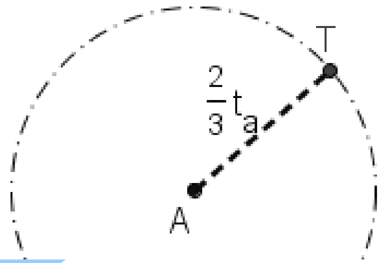

Figure 3: A and T

We know that the distance between any vertex and the centroid T is 2/3 the length of the corresponding median. We use strictly compass and straightedge commands in GeoGebra in this step [FS09].

- We can draw vertex A and centroid T. (see Figure 3).
  - *New point tool* - draw point A.

    T must be a point on the circle centered at A of radius $(2/3)\cdot t_a$
  - *Compass tool* - draw this circle by clicking on left point of t_a and then on the 2/3 point and then on point A.
  - *New point tool* - draw point T by clicking on the circle.
  - *Segment tool* - draw segment AT by clicking on A and then T.

We cannot draw any other object of the triangle (vertex or midpoint) with certainty so we save our worksheet and begin our exploring.

2.3 Exploring with GeoGebra

In this section, we allow ourselves to use all of the tools of GeoGebra while we seek both to understand the construction and find key objects that we **can** construct with just a compass and straightedge.

We do not know the position of the base of the triangle so we draw a dynamic base line (or ray) through A with a "free" point E. (see Figure 4)

- *Line tool* – click on A and then on new point E below and right.

The position of this line can be changed by using the *Move tool* and clicking and dragging the free point E.

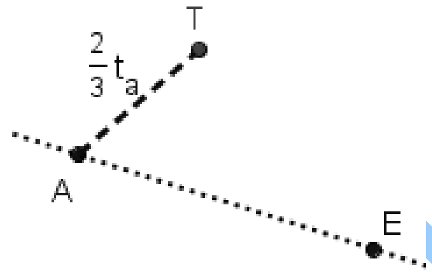

Figure 4. Dynamic base of triangle

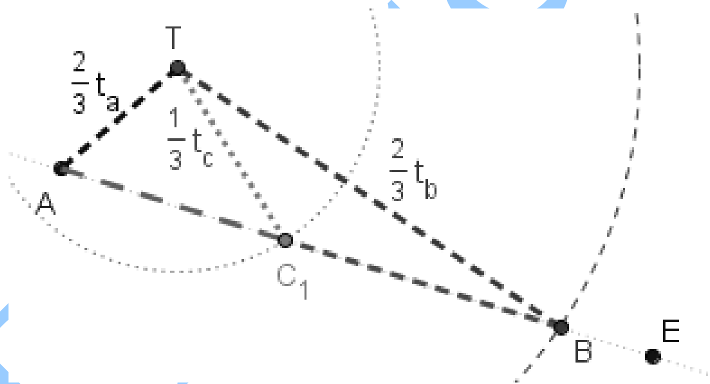

Figure 5. Working points B and C_1

Both the vertex point B and the midpoint C_1 must be on this base line and we know how far both of these are from T.

- On this "dynamic" baseline, we draw B (see Figure 5).
  - *Compass tool* - draw a circle of radius $2/3 \cdot t_b$ using points on t_b (see Figure 2) and centered at T.
  - *Intersection tool* – click on intersection of line AE and this circle.
  - *Segment tool* – click on A and then B to get segment AB.
- On this "dynamic" baseline, we draw C_1 (see Figure 5).
  - *Compass tool* - draw a circle of radius $2/3 \cdot t_c$ centered at T.

- Intersection of AE and this circle.
- Segments AC_1 and C_1B.

Now remember – we are just exploring. This is not a construction. Nor is it quite right. How can we move E to make this construction correct?

Critical Pedagogy: By definition of median, C_1 must be the midpoint of AB. For an arbitrary position of E, we see that this is not so. That is, in Figure 5, AC_1 is **not** the same length as C_1B.

We click and drag point E until AC_1 and C_1B are (approximately) the same lengths by looking at the Algebra View (see Figure 6).

Critical Pedagogy: In our exploring, we see this position is unique. This means there will be only one triangle.

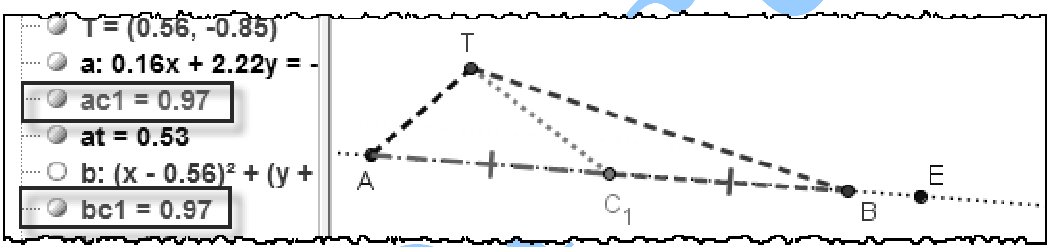

Figure 6. What we want to construct

Here we see that the construction "works". We see the points C_1 and B that we want (but have not yet constructed using compass and straightedge).

Critical Pedagogy: The student must see that if we construct ***either*** of these points, we ***can*** construct the triangle we want.
That is, if we construct C_1, we can then (a) extend along AC_1 *2 times* to get point B and (b) we can extend along C_1T *3 times* to get point A. Alternatively, if we construct B, we can find the midpoint to get C_1 …

Critical Question: Can we construct C_1 or B directly?
Critical Answer: No.

As in a classically taught class, here it is most probable that a little guidance will be needed from the teacher. One guided question is: Can we construct a "helping" point D such that C_1 is the midpoint of the parallelogram ADBT?

- Using the *Parallel tool*, we draw a line through A parallel to TB and using the *Line tool*, we draw a line through TC_1. The intersection point is D. Use Intersect tool to draw D. (See Figure 7.)

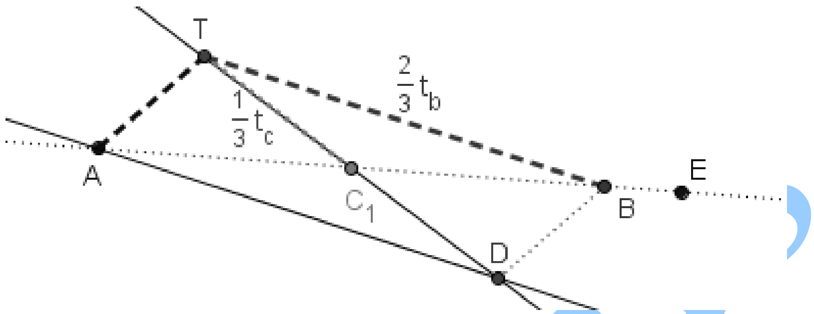

Figure 7. Thinking about how to construct using C+S

Have the students label everything and they should see that:

1. We **can** construct point D. It is the intersection point of
   - a circle centered at A with radius $2/3 \cdot t_b$ and
   - a circle centered at T with radius twice the distance from T to C_1. That is, with radius $2 \cdot (1/3 \cdot t_c)$.
2. C_1 is the **midpoint** of TD.

That is, we can construct D using compass and straightedge and from D, we can construct C_1 which is enough to construct the triangle. Our exploration is done and we are ready to finish our C+S construction.

3 Constructing with GeoGebra

Save your construction as an exploration. Back up to where only A and T are drawn since they were constructed with C+S (Figure 3). Continue with strict compass and straightedge construction in GeoGebra to draw D and C_1.

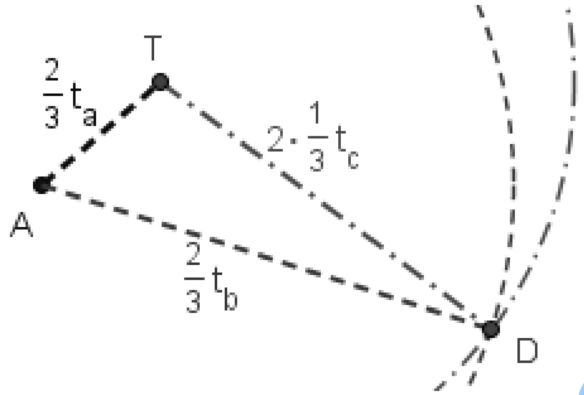

Figure 8. Constructing Point D

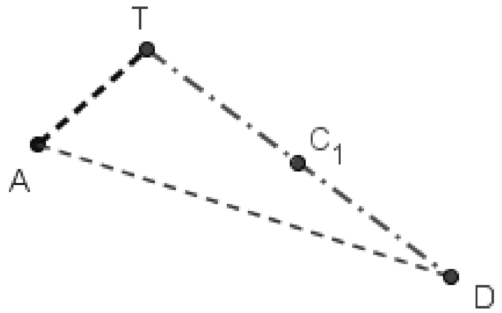

Figure 9. C_1 is Midpoint

Once we have C_1, we can easily construct the rest of the triangle.

4 Testing and Analyzing the Construction

One of the most marvelous things about GeoGebra is that it is dynamic. This means that we can test our construction in several different ways. In this case, because the construction is unique, we test:

- Is the construction stable?
- What are the conditions for the construction to exist?

If a construction is not unique, additional testing is possible.

4.1 Is the construction stable?

Using GeoGebra, we can test that our construction is properly constructed by changing the "givens". Here this means changing the lengths of t_a, t_b and t_c using the *Move tool*. The constructed triangle should change dynamically but always remain true to the goal.

4.2 Does the construction disappear?

Can the students change the givens so that the construction disappears? This means that there are conditions for the existence. This dynamic property of GeoGebra makes it easy (and more interesting) to move on to analysis.

Here simply drag any of the medians to have length longer than the sum of the other 2 medians. What? D and C_1 and the triangle have disappeared (Figure 10 and Figure 11). Now ask why?

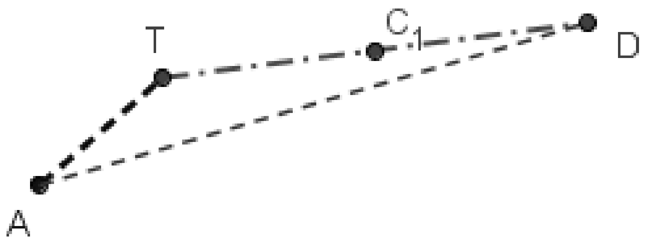

Figure 10. $t_b < t_a + t_c$

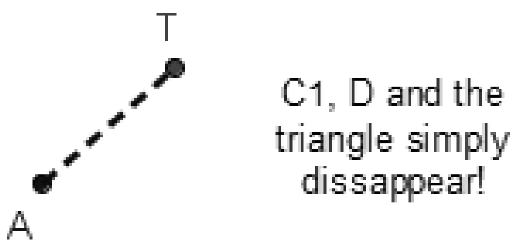

Figure 11. $t_b = t_a + t_c$

The disappearing triangle interests students.

4.3 Final Analysis

Now we can ask - when does a triangle exist? We get them to look in detail at Figure 8, that is, at the construction of triangle ATD. This is just a basic C+S construction of a triangle given 3 side lengths. Here the 3 sides have lengths: $|AT|=2/3\cdot t_a$, $|AD|=2/3\cdot t_b$ and $|TD|=2/3\cdot t_c$.

The students should know that a triangle exists if and only if the sides satisfy the triangle inequality. We write down one inequalities, e.g. $2/3\cdot t_a+ 2/3\cdot t_b \leq 2/3\cdot t_c$. They should see that they can multiply by 3/2 and get the inequality: $t_a+ t_b \leq t_c$. We have them write down the other 2 analogous inequalities and then have them write the concluding sentence:

A triangle with medians $t_a+ t_b$ and t_c exists and is unique if and only if the 3 medians satisfy the triangle inequality.

5 Classroom Experiences

This year, two groups, each of 30 students worked on this and other construction projects. One group solved the constructive tasks using GeoGebra as described above and the other solved the tasks in the classical way using an actual compass and straightedge. The students in the two groups were equally knowledgeable in mathematics and in solving constructive tasks. After 8 classes of procedural study, an assessment was made of their understanding. The following results were obtained. In both groups, the percentage of the successfully completed constructive tasks by students was approximately the same - 60% of students resolved the tasks. But, there was a big difference in the number of students who understood the discussion and analysis and correctly wrote down the conditions for existence and number of solutions. Of the group working with GeoGebra who did the construction, 95% had correctly written analyses. On the other hand, of the group who worked in the classical way and who did the construction, only 25% had correctly written analyses. Also, when constructing with GeoGebra, students themselves noticed other dependencies that applied to the constructed figures - specifically they noted that the triangle inequalities must also apply to the medians of a triangle.

Summary and Conclusions

With help of the GeoGebra, students use dynamic exploration to significantly and vividly improve their total understanding of compass and straightedge constructive tasks. They are better able to understand the discussion and analyses and procedure for construction. Therefore, we can say that resolving these tasks using GeoGebra improves the percentage of correctly and completely solved constructive tasks, which is our goal.

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