# GeoGebra in Mathematical Educational Motivation

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**Abstract:** We describe some possibilities of using GeoGebra software environment in mathematical education with respect to new Slovak secondary school curriculum standards. GeoGebra can help teachers to use a class more effectively, to motivate the students and to teach mathematics with understanding. We present method of generating problems, and some other motivational tools, too.
**Keywords:** Computer Based Math Education, Method of Generating Problems, Slovak Curriculum ISCED 2 and 3, Motivation in mathematics education.

## Introduction

In 2008/2009, new curriculum ISCED 2 & 3 was implemented at Slovak secondary schools which reduced the number of mathematics lessons, and even teachings topics. According to national curriculum every school in Slovakia can prepare one itself. There is possible to increase the number of mathematics subsidy, especially via those topics which are not compulsory. In some sense it means that computer based on maths education can be implemented to teaching process. Slovak national project Infovek is appropriate for applying such ideas. GeoGebra software–which is very useful friendly–can help the teachers to implement ICT programme into their teaching.

According to [Kop97] many educators say that the main goals of teaching mathematics are:

- the development of logical thinking;
- the development of creative thinking;
- the development of person autonomous;

- the development of the ability to solve problems.

GeoGebra allows implementation of such goals in mathematics education.

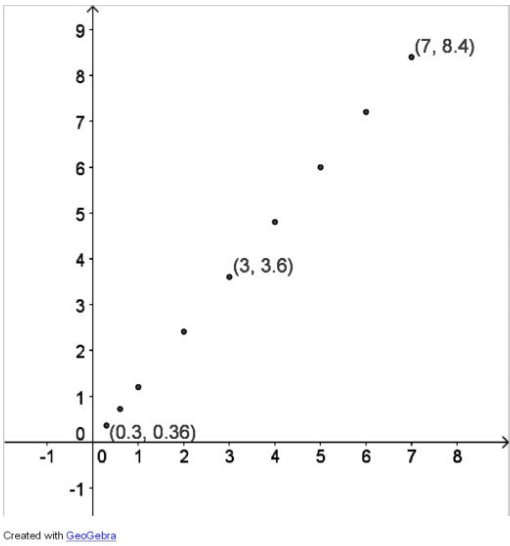
## 1. The process of gaining knowledge and GeoGebra

The process of gaining knowledge in mathematics education by Hejný & Littler ([HL06]) is based on the certain stages. It starts with motivation and its cores are two mental lifts: the first leads from concrete knowledge to generic knowledge, the second one from generic to abstract knowledge. The permanent part of the gaining of knowledge process is crystallisation, i.e. putting new knowledge into the already existing mathematical structure. The whole process has following stages:

1. motivation;
2. isolated models;
3. generic model(s);
4. abstract knowledge;
5. crystallization;
6. automation.

Motivation is the tension which occurs in a person's mind as a result of the discrepancy between the existing and desired states of knowledge. The discrepancy comes from the difference between "I do not know" and "I need to know", or "I can not do that" and "I want to be able to do that", sometimes from other needs and discrepancies, too. GeoGebra can be used as effective motivational tool because it enables to consider many mathematical notions in their mutual relationships, even more visible, and moreover by a dynamical way.

The pupils experiences on some mathematical notion is possible to use as an isolated model. For example, the price of apples due to the their wage is - in GeoGebra - possible represent by the following table and graph prepared (1,20 Euro per 1kg apple). In the first step we represent a function as a set of isolated points (see Figure 1).




| | A | B |
|---|---|---|
| 1 | The wage of apples | Praise in Euro |
| 2 | 0.3 | 0.36 |
| 3 | 0.6 | 0.72 |
| 4 | 1 | 1.2 |
| 5 | 2 | 2.4 |
| 6 | 3 | 3.6 |
| 7 | 4 | 4.8 |
| 8 | 5 | 6 |
| 9 | 6 | 7.2 |
| 10 | 7 | 8.4 |
| 11 | | |
| 12 | | |
| 13 | | |
| 14 | | |

**Figure 1**

This way illustration of dependence is similar to old Babylonian mathematicians who studied the position of planets and represented their movement with the set of isolated points (picture of the positions of planets). In this stage (isolated models) pupils can measure a temperature, a reach of the river in the time etc. GeoGebra allows to put results of their work into table very easy, and then to represent them graphically.

In the third stage (generic models) pupils can try to find the curve passing through some points. It would lead to the graph of the function which represents the concrete relationship. In such case the definition of the function serves as a base for abstract knowledge. Moreover, it is obtained via receiving process spontaneously and naturally, step by step from the previous isolated and generic models. Crystallisation and automation consist from manipulating with concrete functions and even with functions which are not continuous and have more complicated shape.

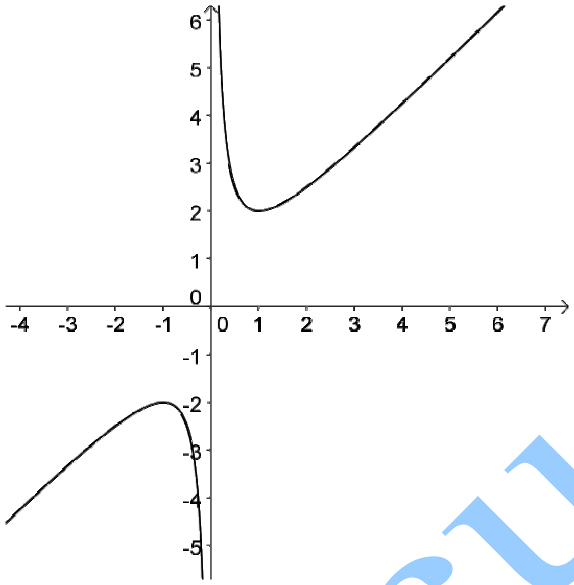



**Figure 2**

## 2. The method of generating problems

The method of generating problems (see [Wit01]) seems to be suitable for this purpose (due to its systematically creating sets of internally connected problems). Student activities and instructions have to be regarded as complementary factors in the learning process. These factors both are necessary and must be systematically related to one another so that optimal progress may occur. The aim of our method is to create areas in which the students may–using the result of guided teaching–move as independently as possible, and in which he/she may develop their own initiatives. The student is considering his own problem and he could ask to assist or help as far as necessary. By this way he can obtain basis for further work. After a problem has been completely solved and clarified the teacher together with students are thinking about further questions and generate problems which are related to the problem just solved. Thus the original problem acts as a generating problem; we will call it generator problem (GP). Related problems are obtained by analogy, variation, generalization, specialization etc. The group of all new problems together with their GP will be called the set of generated problems of the GP or the problem domain of GP.

## 3. The Illustrations

Let us illustrate the method of generating problems via next example. Let us consider an equation of the form $f(x) = f^{-1}(x)$ such that *f* is a function and $f^{-1}$ is its inverse function. Solving such equation provides a student of a secondary school through number varied topics and activities leading to a much better understanding the notions and properties of functions, inverse functions, equations with parameters, and so on. Various computer programs supporting drawing graphs of functions will effectively help in such activities.

- The rational functions of type $\frac{ux+v}{px+r}$

Every function $f(x) = \frac{ux+v}{px+r}$, $x \neq -\frac{r}{p}$ (p, r, u, v are real numbers), can be written in the form $f(x) = a + \frac{k}{x-b}$, $x \neq b$ (a, k, b are real numbers). For this function, the equation $f(x) = f^{-1}(x)$ has for $a \neq 0$ and $b \neq 0$ a form

$$a + \frac{k}{x-b} = b + \frac{k}{x-a} \tag{1}$$

If we solve this equation, we get

$$a + \frac{k}{x-b} = b + \frac{k}{x-a}$$
$$a(x-a)(x-b) + k(x-a) = b(x-a)(x-b) + k(x-b)$$
$$(a-b)(x-a)(x-b) + k(b-a) = 0$$
$$(a-b)((x-a)(x-b) - k) = 0$$

If $a = b$, we get $0 = 0$, and the solution are all real numbers. If $a \neq b$, then

$$(x-a)(x-b) - k = 0$$
$$x^2 + (-a-b)x + (ab-k) = 0$$
$$x_{1,2} = \frac{a+b \pm \sqrt{(a-b)^2 - 4k}}{2}$$

Now, there are three possibilities:

1. If $k > -\frac{1}{4}(a-b)^2$, then the equation (1) has two solutions

$$x_1 = \frac{a+b+\sqrt{(a-b)^2+4k}}{2},\ x_2 = \frac{a+b-\sqrt{(a-b)^2+4k}}{2}.$$

Notice that $(a - b)^2 > 0$ and for every positive number k the equation (1) has two solutions.

2. If $k = -\frac{1}{4}(a-b)^2$, then one solution is $x = \frac{1}{2}(a + b)$.

3. If $k < -\frac{1}{4}(a-b)^2$, then (1) does not have any solution.

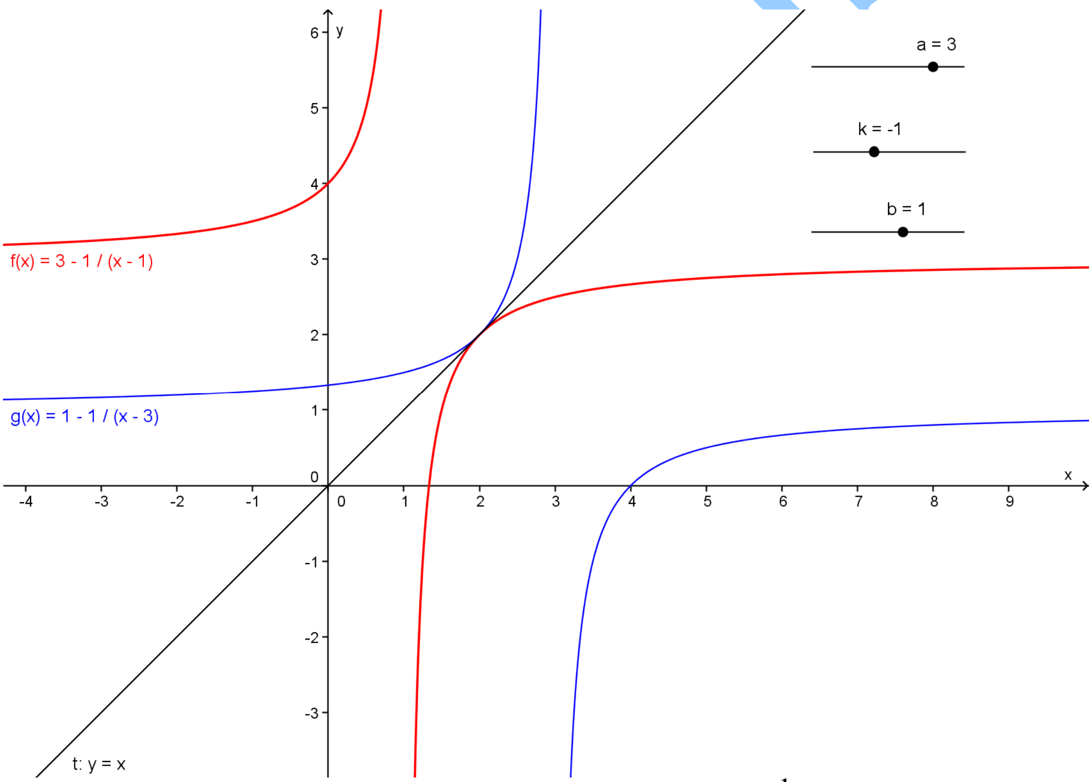



**Figure 3: Rational function f with his inverse function $g = f^{-1}$ in GeoGebra**

The solutions have a geometrical interpretation. Graphs of the functions f and $f^{-1}$ are hyperbolas. The question is how many common points do these hyperbolas have?

In case a = b, the hyperbola, i.e. graph of the function f, is symmetrical with respect to the axis y = x. Therefore the solution of equation (1) is all real numbers.

In case a ≠ b there are three possibilities. First, the hyperbolas have two common points. For every positive number k the hyperbola, i.e. graph of the function f, has two common points with the axis of axial symmetry y = x. These common points are the common points with the hyperbola, i.e. graph of the function $f^{-1}$. Second, one branch of the hyperbola f touches the one branch of the hyperbola $f^{-1}$. They have a common tangent y = x at the common point. This situation we explain by the function $f(x) = 3 - \frac{1}{x-1}$ (see Figure 3). Third, the hyperbolas do not have any common point.

- The quadratic function of type $x^2 + a$

The equation $f(x) = f^{-1}(x)$ has for quadratic function of type $x^2 + a$ and $x \geq 0$ for real parameter a the form

$$x^2 + a = \sqrt{x - a} \tag{2}$$

If we solve this equation, we get

$$\begin{aligned}
&x^4 + 2ax^2 + a^2 = x - a \\
&x^4 + 2ax^2 - x + a^2 + a = 0 \\
&x^4 + 2ax^2 + x^2 - x^2 - x + a^2 + a = 0 \\
&x^4 + (2a+1)x^2 - x^2 - x + a^2 + a = 0 \\
&\left(x^2 + \left(a + \frac{1}{2}\right)\right)^2 - \left(a + \frac{1}{2}\right)^2 - x^2 - x + a^2 + a = 0 \\
&\left(x^2 + \left(a + \frac{1}{2}\right)\right)^2 - \left(x + \frac{1}{2}\right)^2 = 0 \\
&(x^2 + x + (a+1))(x^2 - x + a) = 0
\end{aligned}$$

That means $x^2 + x + (a+1) = 0$ or $x^2 - x + a = 0$. If we solve these last two quadratic equations, then we received

$$x_{1,2} = \frac{-1 \pm \sqrt{1 - 4(a+1)}}{2}, \quad x_{3,4} = \frac{1 \pm \sqrt{1 - 4a}}{2}.$$

There is not all solutions are correct in the set of real numbers, because in the beginning we start with non-equivalent operation. It is possible to show that following solutions in the table are correct. By the graphical representation of the equation can help us also GeoGebra (see Figure 4).

| | |
|---|---|
| $a\in(-\infty, 0)$ | $x=\frac{1+\sqrt{1-4a}}{2}$ |
| $a\in\left\langle 0,\frac{1}{4}\right)$ | $x_{1,2}=\frac{1\pm\sqrt{1-4a}}{2}$ |
| $a=\frac{1}{4}$ | $x=\frac{1}{2}$ |
| $a\in\left(\frac{1}{4},\infty\right)$ | Not any solution. |

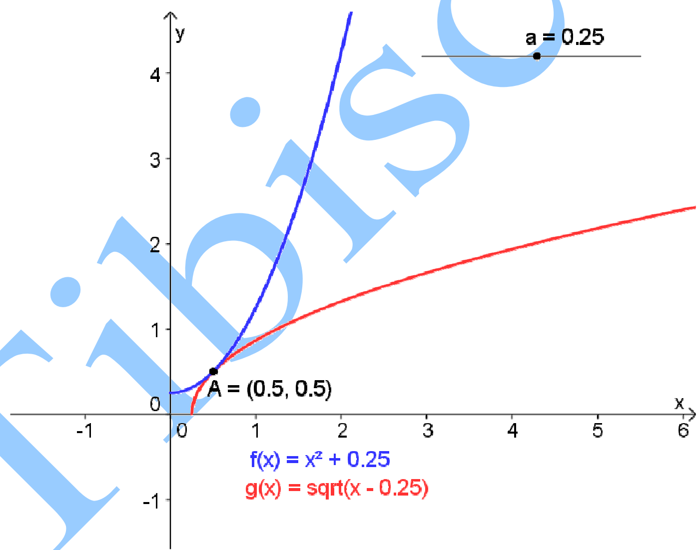



**Figure 4: Function f with his inverse function $g = f^{-1}$ in GeoGebra**

## Conclusion

Our examples illustrate how it is possible to use Geogebra in the development of the process of gaining mathematical knowledge through education and also the method of generating problems with generator problem. Benefit of this purpose is obtained from connections between teaching mathematical analysis and both analytical and synthetic geometry at school. It is very important that the students' knowledge be not isolated.

A teacher has a possibility to explain the notions of inverse function, graphs of different types of functions. Students can easily see fact that the graphs of the function f and function $f^{-1}$ are symmetrical with respect to the axis y = x. In this situation GeoGebra can serve as useful tool for drawing those graphs. Student can also utilize it by finding the numerical solutions the equations $f(x) = f^{-1}(x)$ for some types of functions (trigonometric, exponential functions, etc.)

The method of generating problems could be useful even in other parts of school mathematics. Another applications reader can find in [Bil10], [H+09], [Tak09] and [Tka07].

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