Maximal elements and their group-like set

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ABSTRACT. This paper examines an example of a set *Q* with a partial product that is not a presemigroup (i.e. a set with a partial product that becomes a semigroup with zero when a zero is adjoined and all undefined products are defined to be zero). Despite this shortcoming, the non-associative product has succeeded in creating an "F*-inverse" theory on *Q*. The group-like set of maximal elements of *Q* is isomorphic to the group-like set of integers.
KEYWORDS: inverse semigroup, F*-inverse, maximal element, group-like set

Introduction

If *S* is a semigroup with zero then $S^* = S - \{0\}$ has a partial binary operation induced by the semigroup product. The set S^* equipped with this partial binary operation is called a presemigroup (if for any non-zero elements $x, y \in S$ we have $xy \neq 0$ then the presemigroup S^* is just a semigroup). Now let *P* be a set equipped with a partial product. When a 0 is adjoined and all undefined product are defined to be zero then the set $P^0 = P \cup \{0\}$ become a set equipped with a binary operation. This binary operation can be associative or not. In the first case *P* is a presemigroup, and the semigroup P^0 can be, in particular, an inverse semigroup (with zero), furthermore an E*-unitary or an F*-inverse semigroup. Thus, in this case we can develop E*-unitary or F*-inverse theory on the presemigroup *P*. A natural question arises here: is it possible to develop a such theory on a set *Q* which is equipped with a partial binary operation which is not a presemigroup? Our example, in this paper, is a set Q equipped with a non-associative partial binary operation. All its properties convincingly show that the answer is yes to the question above. We can say that Q is an F*-inverse semigroup-like set. In our example the F*-inverse properties leads exactly to the group-like set of integers as the group-like set of maximal elements of Q.

1. Background

An inverse semigroup S is a semigroup in which every element x has a unique inverse x^{-1} in the sense that $x = xx^{-1}x$ and $x^{-1}xx^{-1} = x^{-1}$. The semigroup S is inverse if and only if it is regular (i.e. for each element x there exists an element y such that $xyx = x$) and its idempotents commute.

The natural partial order on S defined by $x \le y \Leftrightarrow x = xx^{-1}y$ (equivalently, $x \le y \Leftrightarrow x = yx^{-1}x$) plays a central role in the theory of inverse semigroups. An inverse semigroup with zero is E*-unitary when any element above a non-zero idempotent is itself an idempotent. An inverse semigroup with zero is F*-inverse when any non-zero element is beneath a unique maximal element. M. Szendrei [Sze87] introduced the class of E*-unitary inverse semigroups, and A. Nica [Nic94] the class of F*-inverse semigroups. Any F*-inverse semigroup is E*-unitary.

Let S be an F*-inverse monoid, M the set of all maximal elements of S, and $D = \{(x, y) \in M \times M \mid xy \ne 0\}$. A partial binary operation $\circ$ on the set M with domain D is defined as follows:

$$(x, y) \in D; \qquad x \circ y = \langle xy \rangle,$$

where $\langle xy \rangle$ is the unique maximal element above xy. The set of maximal elements M with domain D and the above partial binary operation is a group-like set.

Following Jekel [Jek77] and Lawson [Law02], a group-like set P with domain $D \subset P \times P$ is a set equipped with a partial binary operation $m : D \to P$ (we will denote $m(x, y)$ by $x \circ y$), with a distinguished element 1, and an involution $x \mapsto x^{-1}$ ($(x^{-1})^{-1} = x$) satisfying the following axioms:

(P_1) $(x,1),(1,x) \in D$ and $1 \circ x = x \circ 1 = x$ for all $x \in P$;

(P_2) $(x, x^{-1}),(x^{-1}, x) \in D$ and $x \circ x^{-1} = x^{-1} \circ x = 1$ for all $x \in P$;

(P_3) if $(x, y) \in D$ then $(y^{-1}, x^{-1}) \in D$ and $(x \circ y)^{-1} = y^{-1} \circ x^{-1}$.

Let (P,D) and (P',D') be two group-like sets. A group-like set isomorphism is a bijective map $\varphi : P \to P'$ such that: (1) $(x,y) \in D \Leftrightarrow (\varphi(x), \varphi(y)) \in D'$; (2) $\varphi(x \circ y) = \varphi(x) \circ \varphi(y)$ for all $(x,y) \in D$; (3) $\varphi(x^{-1}) = [\varphi(x)]^{-1}$ for all $x \in P$.

An example of a group-like set is the group-like set of integers (Z,D) where $D = \{(x,y) \in Z \times Z \mid xy \leq 0\}$, the partial "multiplication" (with domain *D*) is the usual addition, 0 is the distinguished element, and $x \mapsto -x$ is the involution. We can see that $(3+(-5))+7=5$ but $3+(-5+7)$ is not defined and therefore $(3+(-5))+7 \neq 3+(-5+7)$. Thus our example cannot be converted into a semigroup (with zero) by adjoining a zero such that $x \circ y = 0$ if $x \circ y$ is undefined and $0 \circ x = x \circ 0 = 0 \circ 0 = 0$.

2. The inverse-semigroup properties of a non-associative partial multiplication

Let *N* be the set of non-negative integers and

$$Q = \{(a,b,m) \in N^3 \mid a,b \leq m\}$$

Define a partial multiplication on *Q* by:

$$(a,b,m)\cdot(a',b',m') = \begin{cases} (a-b+a',b',m') & \text{if } m \leq m',\ b \leq a' \text{ and } a'-b \leq m'-m \\ (a,b-a'+b',m) & \text{if } m \geq m',\ b \geq a' \text{ and } b-a' \leq m-m' \end{cases}$$

Remarks. This partial multiplication is not associative.
For example, if $x=(2,4,4)$, $y=(3,2,6)$, $z=(7,8,20)$ then $x\cdot(y\cdot z)=(6,8,20)$ but $(x\cdot y)\cdot z$ is undefined since $x\cdot y$ is undefined. For this reason, *Q* is not converted into a semigroup (with zero) by adjoining a zero such that $x\cdot y=0$ if $x\cdot y$ is undefined and $0\cdot x = x\cdot 0 = 0\cdot 0 = 0$. Although $(Q,\cdot)$ is not a presemigroup, below we will inspect the "inverse-semigroup" properties of $(Q,\cdot)$.

Proposition 1. *The set*

$$E(Q) = \{(a,a,m) \in Q\}$$

is the set of idempotents of $(Q,\cdot)$ *and* $x=(a,b,m) \mapsto x^{-1}=(b,a,m)$ *is an involution on* $(Q,\cdot)$, *that is*:

$$(x^{-1})^{-1} = x \quad ; \quad \text{if } \exists x\cdot y \text{ then } \exists y^{-1}\cdot x^{-1} \text{ and } (x\cdot y)^{-1} = y^{-1}\cdot x^{-1}.$$

Proposition 2. *If* $x=(a,b,m)$ *then*

$$\exists x^{-1}\cdot x,\ \exists x\cdot x^{-1}\ \textit{and}\ x^{-1}\cdot x=(b,b,m),\ x\cdot x^{-1}=(a,a,m).$$

Moreover:

$$\exists x\cdot(x^{-1}\cdot x),\ \exists(x\cdot x^{-1})\cdot x\ \textit{and}\ x\cdot(x^{-1}\cdot x)=(x\cdot x^{-1})\cdot x=x$$

Proposition 3. *If* $e,f\in E(Q)$ *such that* $\exists e\cdot f$ *then* $\exists f\cdot e$ *and* $e\cdot f=f\cdot e\in E(Q)$.

Proposition 4. *If* $x,y\in Q$ *such that* $x^{-1}\cdot x=y\cdot y^{-1}$ *then* $\exists x\cdot y$ (*called the restricted multiplication*)

Proof. If $x=(a,b,m),\ y=(a',b',m')\in Q$ such that $x^{-1}\cdot x=y\cdot y^{-1}$ then

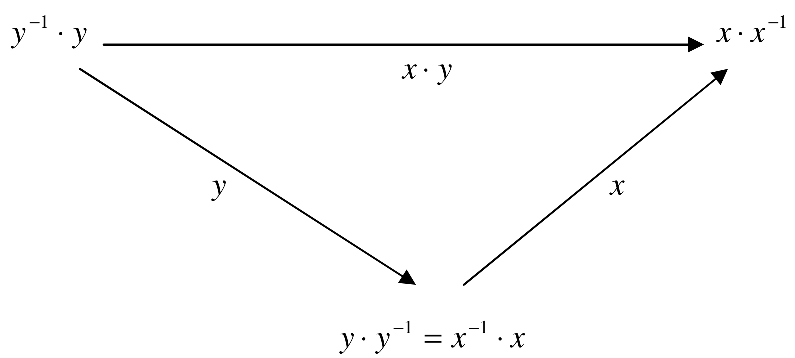
$$b=a'\quad \textit{and}\quad m=m'.$$

Hence

$$x\cdot y=(a,b,m)\cdot(b,b',m)=(a,b',m).$$

Proposition 5. *Let G be the category defined by*

- $ObG=E(Q)$
- $Hom_G(e,f)=\{x\in Q\mid x^{-1}\cdot x=e\ \textit{and}\ x\cdot x^{-1}=f\}$
- *Composition of morphisms is given by restricted multiplication*:



Then G is a groupoid (*the associated groupoid*).

(It is clear that if $x=(a,b,m),\ y=(a',b',m')\in Q$ such that $x^{-1}\cdot x=y\cdot y^{-1}$ then

$$(x\cdot y)^{-1}\cdot(x\cdot y)=(b',a,m)\cdot(a,b',m)=(b',b',m)=y^{-1}\cdot y \quad and$$
$$(x\cdot y)\cdot(x\cdot y)^{-1}=(a,b',m)\cdot(b',a,m)=(a,a,m)=x\cdot x^{-1}.)$$

Remark. Here the restricted multiplication is associative: if $x=(a,b,m)$, $y=(a',b',m')$, $z=(a'',b'',m'')$ $(x,y,z\in Q)$ such that $x^{-1}\cdot x=y\cdot y^{-1}$ and $y^{-1}\cdot y=z\cdot z^{-1}$ then

$$b=a',\quad m=m',\quad b'=a''\quad m'=m''$$

and

$$(x\cdot y)\cdot z=(a,b',m)\cdot(b',b'',m)=(a,b'',m),$$
$$x\cdot(y\cdot z)=(a,b,m)\cdot(b,b'',m)=(a,b'',m).$$

Now, we define the relation $\le$ on Q as follows. If $x=(a,b,m),\ y=(a',b',m')\in Q$, then

$$x\le y \quad\Leftrightarrow\quad m\ge m',\ a\ge a'\ and\ a-a'=b-b'\le m-m'$$

It is straightforward to see that this is a partial order on Q.

Proposition 6. *The following are equivalent:*
(1) $x\le y$;
(2) $\exists(x\cdot x^{-1})\cdot y\ and\ (x\cdot x^{-1})\cdot y=x$;
(3) $\exists y\cdot(x^{-1}\cdot x)\ and\ y\cdot(x^{-1}\cdot x)=x.$

Proof. Let $x=(a,b,m),\ y=(a',b',m')\in Q$.
(1) $\Rightarrow$(2) Since $x\le y$ and $x\cdot x^{-1}=(a,a,m)$, it follows that

$$(x\cdot x^{-1})\cdot y=(a,a,m)\cdot(a',b',m')=(a,a-a'+b',m)=(a,b,m)=x.$$

(2) $\Rightarrow$(1) Since

$$(x\cdot x^{-1})\cdot y=(a,a,m)\cdot(a',b',m')=$$
$$=\begin{cases}(a-a+a',b',m') & if\ m\le m',\ a\le a'\ and\ a'-a\le m'-m\\ (a,a-a'+b',m) & if\ m\ge m',\ a\ge a'\ and\ a-a'\le m-m',\end{cases}$$

it follows that $(x\cdot x^{-1})\cdot y=x$ implies
$m\ge m',\ a\ge a'\ and\ a-a'=b-b'\le m-m'$;
that is $x\le y$.

(1) $\Leftrightarrow$ (3) Similarly.

Proposition 7. *Let*

$$M_1 = \{(m,0,m) \in Q \mid m \in N\} \text{ and } M_2 = \{(0,m,m) \in Q \mid m \in N\}.$$

Then

$$M = M_1 \cup M_2$$

is the set of all maximal elements of Q.

Proof. If $(m,0,m) \le (a',b',m')$ then $m \ge m', m \ge a'$ and $m - a' = 0 - b' \le m - m'$. It follows that $b' = 0, a' = m$ and $m' = m$. Hence $(m,0,m)$ is a maximal element of Q.

If $(0,m,m) \le (a',b',m')$ then $m \ge m', 0 \ge a'$ ($\Rightarrow a' = 0$) and $0 - 0 = m - b' \le m - m'$. Thus $b' = m$. Since $b' \le m'$ it follows that $m' = m$. Consequently, $(0,m,m)$ is also a maximal element of Q.

Let $(a,b,m) \in Q$ such that $a,b > 0$. Then $(a,b,m) < (a-1,b-1,m-1)$ since $m > m-1, a > a-1$ and $a-(a-1) = b-(b-1) \le m-(m-1)$. Now, if $a = 0$ and $b < m$ then $(0,b,m) < (0,b,m-1)$ since $m > m-1,\ 0 \le 0$ and $0 - 0 = b - b < m-(m-1)$. Analogously, $(a,0,m) < (a,0,m-1)$ if $a < m$. So, the proof of Proposition 7 is complete.

Proposition 8. *If* $x = (a,b,m) \in Q$, *then the unique maximal element above x is* $(a-b,0,a-b)$ *if* $a \ge b$, *and it is* $(0,b-a,b-a)$ *if* $a \le b$.

Proof. If $a \ge b$ then $(a,b,m) \le (a-b,0,a-b)$, since $m \ge a-b, a \ge a-b$ and $a-(a-b) = b - 0 \le m-(a-b)$.

Analogously, $(a,b,m) \le (0,b-a,b-a)$ if $a \le b$.

Now, if $(a,b,m) \le (m',0,m')$, that is $m \ge m', a \ge m'$ and $a - m' = b - 0 \le m - m'$, then $m' = a - b$ and $a \ge b$. So in the case $a \ge b$, $(a-b,0,a-b)$ is the unique maximal element which is above (a,b,m). Analogously, if $a \le b$ then $(0,b-a,b-a)$ is the unique maximal element such that $(a,b,m) \le (0,b-a,b-a)$.

Remarks. Taking into account the definition of F*-inverse semigroup, we can say that Q is an F*-inverse semigroup-like set. Also, since $(a,a,m) \le (a',b',m')$ implies $a' = b'$ we can say that Q is an E*-unitary inverse semigroup-like set.

Proposition 9. *Let* $x, y \in M$ *. Then* $\exists x \cdot y$ *if and only if*

$$(x, y) \in (M_1 \times M_2) \cup (M_2 \times M_1).$$

Proof. Let $(m,0,m) \in M_1$ and $(0,m',m') \in M_2$. Then,

$$(m,0,m)\cdot(0,m',m') = \begin{cases} (m,m',m') & \text{if } m \le m' \\ (m,m',m) & \text{if } m \ge m' \end{cases}$$

and

$$(0,m',m')\cdot(m,0,m) = \begin{cases} (m-m',0,m) & \text{if } m' \le m \\ (0,m'-m,m') & \text{if } m' \ge m \end{cases}$$

Conversely, let $x, y \in M$ such that $\exists x \cdot y$. If both x and y belong to M_1 then follows $x = (0,0,0)$ or $y = (0,0,0)$. The same conclusion is obtained if $x, y \in M_2$. Since $M_1 \cap M_2 = \{(0,0,0)\}$, the proof of Proposition 9 is complete.

Corollary 10

(1) *If* $m \ge m'$ *then* $(m-m',0,m-m')$ *is the unique maximal element above* $(m,0,m)\cdot(0,m',m')$ *and it is the unique maximal element above* $(0,m',m')\cdot(m,0,m)$.

(2) *If* $m \le m'$ *then* $(0,m'-m,m'-m)$ *is the unique maximal element above* $(m,0,m)\cdot(0,m',m')$ *and it is the unique maximal element above* $(0,m',m')\cdot(m,0,m)$.

A partial binary operation $\circ$ on the set M with domain $D = \{(x, y) \in M \times M \mid \exists x \cdot y\}$ is defined as follows:

$$(x, y) \in D; \qquad x \circ y = \langle x \cdot y \rangle,$$

where $\langle x \cdot y \rangle$ is the unique maximal element above $x \cdot y$. By Proposition 9,

$$D = (M_1 \times M_2) \cup (M_2 \times M_1)$$

and

$$(m,0,m)\circ(0,m',m') = (0,m',m')\circ(m,0,m) = \langle (m,0,m)\cdot(0,m',m') \rangle =$$
$$= \langle 0,m',m')\cdot(m,0,m) \rangle = \begin{cases} (m-m',0,m-m') & \text{if } m \ge m' \\ (0,m'-m,m'-m) & \text{if } m \le m' \end{cases}$$

Proposition 11. *The set M of maximal elements of Q with domain D equipped with the above partial binary operation $\circ$, with the distinguished element $1=(0,0,0)$ and the involution $x \mapsto x^{-1}$, is a group-like set isomorphic to the group-like set of integers.*

Proof. The first part of this Proposition can be obtained by routine verifications taking into account the definition of group-like set. It is straightforward to see that $\varphi : M \to Z$ defined by

$$\varphi((a,b,m)) = \begin{cases} a & if\ a = m\ and\ b = 0 \\ -b & if\ b = m\ and\ a = 0 \end{cases}$$

is an isomorphism of group-like sets.

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