On The Algebraic Modeling of Planar Robots

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ABSTRACT: In the last twenty five years, the technological world has been witnessing mammoth surge of activity in robotics, both at research level as well as in terms of capturing the attention of the general public. Many of these robots have serial mechanical architecture starting from a fixed base and each link is connected through a joint to the next link in the chain. Various methods from affine algebra have been used for modeling such robots especially for solving kinematic problems. In this paper, we introduce and discuss the algebraic modeling of a planar robot with four revolute joints and one prismatic joint and introduce a new concept of operational level space of a robot.
KEYWORDS: Robot, Joint space, Configuration space, Revolute joints, Prismatic Joints.

Introduction

A robot is a mechanical intelligent agent which can perform tasks on its own, or with guidance. In practice a robot is usually an electro-mechanical machine which is guided by computer and electronic programming. The word robot was introduced in 1921 by the Czech playwright Karel Capek in his satirical play R. U. R. (Rossum's Universal Robots), where he depicted robots as machines which resembled people but worked tirelessly. The early work leading up to today's robots began after World War II in the development of remotely controlled mechanical manipulators for handling radioactive material. These early mechanisms were of the master-slave mechanism.

In parallel with this was the development of Computer Numerically Controlled (CNC) machine tools for accurate milling of low-volume, highperformance aircraft parts. Later, robots were developed in which the key innovation was programmability, it could be retooled and reprogrammed at relatively low cost so as to enable it to perform a wide variety of tasks. Tomovic and Boni [KW09] developed a pressure sensor for the robot which enabled it to squeeze on a grasped object and then develop one of two different grasp patterns. At about the same time, a binary robot vision system which enabled the robot to respond to obstacles in its environment was developed by McCarthy and colleagues in 1963. Many other kinematic models for robot arms, such as the Stanford manipulator, the Boston arm, the AMF (American Machine and Foundry) arm, and the Edinburgh arm, were also introduced around this time.

In the 1980s, many efforts were made to improve the performance of industrial robots in fine manipulation tasks: active methods using feedback control to improve positioning accuracy and program compliance, and passive methods involving a mechanical redesign of the arm. The trend in the nineties has been towards robots that are modifiable for different assembly operations. One such robot is called Robotworld, manufactured by Automatix, which features several four degree of freedom modules suspended on air bearings from the stator of a Sawyer effect motor. By attaching different end-effectors to the ends of the modules, the modules can be modified for the assembly task at hand. It is routine to see robot manipulators being used for welding and painting car bodies on assembly lines, stuffing printed circuit boards with IC components. So, it is very important to be capable of moving wherever it needs to. More information can be sought in [KW09].

The relation between algebra and robotics is very well explained in [CLO97]. One interesting problem in robotics is the forward kinematics in which algebraic approach has been tried by many scientists. This paper presents the algebraic modeling of robots and its topological equivalence with the geometric objects. This concept of modeling robots through the smaller entities than the operational space of the robot is first time introduced in the robotics literature.

1. Kinematic analysis

A robot manipulator is composed of a set of links connected together by various joints. The joints can either be very simple, such as a revolute joint or a prismatic joint, or else they can be more complex, such as a ball and socket joint. A revolute joint is like a hinge and allows a relative rotation about a single axis, and a prismatic joint permits a linear motion along a single axis, namely an extension or retraction. The difference between the two situations is that, in the first instance, the joint has only a single degree-of-freedom of motion: the angle of rotation in the case of a revolute joint, and the amount of linear displacement in the case of a prismatic joint. In this article it is assumed throughout that all joints have only a single degree-of-freedom. Note that the assumption does not involve any real loss of generality, since joints such as a ball and socket joint (two degrees-of-freedom) or a spherical wrist (three degrees-of-freedom) can always be thought of as a succession of single degree-of-freedom joints with links of length zero in between. With the assumption that each joint has a single degree-of-freedom, the action of each joint can be described by a single real number: the angle of rotation in the case of a revolute joint or the displacement in the case of a prismatic joint. The objective of forward kinematic analysis is to determine the cumulative effect of the entire set of joint variables. We are going to develop a set of steps that provide a systematic procedure for performing the analysis of planar robotic motion. However, the kinematic analysis of an n-link manipulator can be more complex and the conventions introduced below simplify the analysis considerably.

A robot manipulator with n joints will have n + 1 links, since each joint connects two links. We number the joints from 1 to n, and we number the links from 0 to n, starting from the base. By this convention, joint i connects link i − 1 to link i. We will consider the location of joint i to be fixed with respect to link i − 1. When joint i is actuated, link i moves. Therefore, link 0 (the first link) is fixed, and does not move when the joints are actuated. Of course the robot manipulator could itself be mobile (e.g., it could be mounted on a mobile platform or on an autonomous vehicle), but we will not consider this case in the present paper, since it can be handled easily by slightly extending the techniques presented here. With the i^{th} joint, we associate a joint variable, denoted by l_i. In the case of a revolute joint, $l_i = \theta_i$ is the angle of rotation, and in the case of a prismatic joint, l_i is the joint displacement. To perform the kinematic analysis, we rigidly attach a coordinate frame to each link. This means that, whatever motion the robot executes, the coordinates of each point on link i are constant when expressed in the i^{th} coordinate frame. Furthermore, when joint i is actuated, link i and its attached frame experience a resulting motion. The frame which is attached to the robot base is referred to as the inertial frame. Now suppose A_i is the homogeneous transformation matrix that expresses the position and orientation of the coordinate system. The matrix A_i is not constant, but varies as the configuration of the robot is changed. However, the assumption that all joints are either revolute or prismatic means that A_i is a function of only a single joint variable, namely l_i. This is what the forward kinematics is. For the basic understanding of the kinematic problems one can refer [CLO97]. Thus it is possible to achieve a considerable amount of streamlining and simplification by introducing further conventions, which is explained in the later part. In fact we will be more specific with the robot structure and step by step modeling process is described in the next section.

2. Algebraic modeling

In this paper, we introduce and describe the algebraic modeling of a new combination of arm joints. Here we deal with a planar robot with 4 revolute joints and 1 prismatic joint. Consider a planar robot with a revolute joint 1, segment s_2 of length l_2, a revolute joint 2, segment s_3 of length l_3, a prismatic joint 3 with settings, length of segment s_4 is l_4 ε [0, m_4], a revolute joint 4, segment s_5 of length l_5 and finally a revolute joint 5 with segment 6 being the hand of the robot. We assume that the base segment's length is greater than the sum of the lengths of the other segments.

First, we find the joint and configuration spaces *J* and *C* for this particular robot. As discussed earlier, the first, second, fourth and fifth joints are revolute. As we know, the position or setting of a revolute joint between segments i and i+1 is described by angle θ_i (counter clockwise) from segment i to segment i+1. Thus the totality of settings of such a joint can be parameterized by a circle S or by the interval [0, 2π] with the end points identified. Therefore, there is S = [0, 2π] factor for each of the 4 revolute joints. The setting of the joint 3 which is prismatic in nature is described by an interval [0, m_4] over which l_4 ranges. Therefore the joint space is *J* is S×S×S×S×I where the interval I is [0, m_4].

The configuration space *C* is given by *C* =*U*× *V* where *V* = S describes the possible description of the hand and *U*, a subset of the plane R^2 which describes the possible location of the revolute joint 5. If the prismatic joint is fully retracted, one can observe that this joint can trace out a circle of radius l_3 about the revolute joint 2. When the prismatic joint is fully extended, the joint can trace out a circle of radius l_3+ m_4 about the revolute joint 2. Any point within the annulus of inner radius l_3 and outer radius l_3+ m_4 can be attained by extending the prismatic joint to the required length and rotating it by the appropriate angle. We assume that the points do not lie below the floor to which the base arm of the robot is attached.

Therefore *U* consists of the annulus excluding the part lying below the floor. We take the origin of the coordinate system to be the revolute joint 2. We can represent *U* as

$$U = \{(x, y) \text{ in } R^2 \mid l_3 \leq x^2 + y^2 \leq l_3 + m_4 \}$$

Now we proceed towards constructing an explicit formula for the wrapping f: *J* -→ *C* in the form of the trigonometric function of the joint angles. Let us consider a co ordinate system (x_1, y_1) at revolute joint 1 where x_1 axis is parallel to the horizontal line, another co ordinate system (x_2, y_2) at revolute joint 2 where x_2 axis is along segment 2, another co ordinate system (x_3, y_3) at revolute joint 3 where x_3 axis is along segment 3, another co ordinate system (x_4, y_4) at revolute joint 4 where x_4 axis is along segment 5. Angles θ_i's represent the same as discussed earlier. The transformation matrix A_1 is the matrix for the rotation θ_1 as shown below:

$$A_1 = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & 0 \\ \sin\theta_1 & \cos\theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 & 0 \\ \sin\theta_2 & \cos\theta_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

As the joint 3 is prismatic, there is no rotation about this joint. The transformation matrix A_3 translates the co ordinates (l_3, 0) as shown below:

$$A_3 = \begin{bmatrix} 1 & 0 & l_3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Joint 4 is revolute and segment 4 can have length l_4 in [0. m_4]. The transformation matrix A_4 rotates by angle θ_4 and translates by (l_4, 0), as given below in figure1:

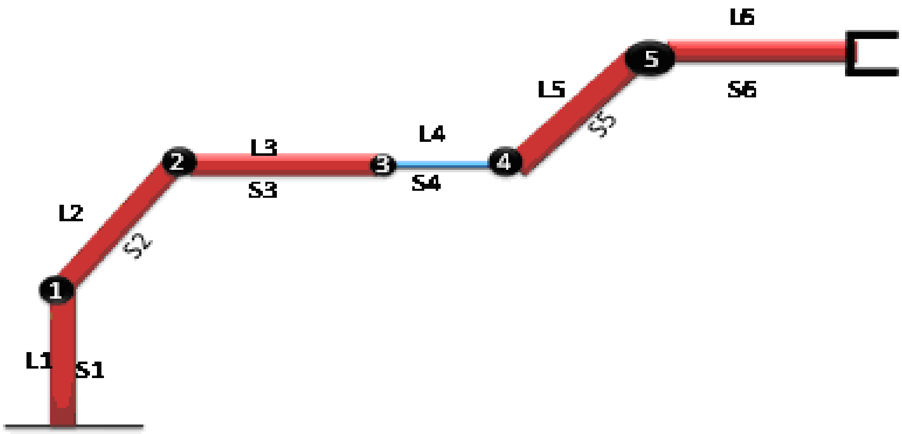

Figure 1. A six arm linkage with 4 revolute joints and 1 prismatic joint

Also,

$$A_4 = \begin{bmatrix} \cos\theta_4 & -\sin\theta_4 & 0 \\ \sin\theta_4 & \cos\theta_4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Joint 5 is also revolute. The transformation matrix A_5 is nothing but the matrix rotation by the angle θ_5 as shown below:

$$A_5 = \begin{bmatrix} \cos\theta_5 & -\sin\theta_5 & 0 \\ \sin\theta_5 & \cos\theta_5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Now we can set the hand position using the following expression:

$$\begin{bmatrix} x_1 \\ y_1 \\ 1 \end{bmatrix} = A_1\ A_2\ A_3\ A_4\ A_5 \begin{bmatrix} x_5 \\ y_5 \\ 1 \end{bmatrix}$$

where the product $A_1A_2A_3A_4A_5$ has the following form:

$$A_1\ A_2\ A_3\ A_4\ A_5 = \begin{bmatrix} \cos(\theta_1+\theta_2+\theta_4+\theta_5) & -\sin(\theta_1+\theta_2+\theta_4+\theta_5) & (l_3+l_4)\cos\theta_1 \\ \sin(\theta_1+\theta_2+\theta_4+\theta_5) & \cos(\theta_1+\theta_2+\theta_4+\theta_5) & (l_3+l_4)\sin\theta_1 \\ 0 & 0 & 1 \end{bmatrix}$$

Since the hand is attached directly to revolute joint 5, the (x_1, y_1) co ordinate of the hand can be obtained from the (x_5, y_5) co-ordinates by substituting $x_5 = y_5 = 0$. Using the transformation matrices:

$$\begin{bmatrix} x_1 \\ y_1 \\ 1 \end{bmatrix} = A_1\ A_2\ A_3\ A_4\ A_5 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

we can now determine the position of the hand from θ_1, θ_5 and l_5. We use the direction of the x_5 axis to specify the orientation of the hand. The angle between the x_1 and the x_5 axis is $\theta_1 + \theta_2 + \theta_4 + \theta_5$.
Then the mapping f: *J* -> C is given by

$$f(\theta_1, \theta_2, \theta_4, \theta_5, l_4) = \begin{bmatrix} (l_3 + l_4)\cos\theta_1 \\ (l_3 + l_4)\sin\theta_1 \\ (\theta_1 + \theta_2 + \theta_4 + \theta_5) \end{bmatrix} \quad (1)$$

Note that $x_1 = (l_3+l_4)\cos\theta_1$ and $x_2 = (l_3+l_4)\sin\theta_1$ are the co-ordinates of the position of the hand and $\theta_1 + \theta_2 + \theta_4 + \theta_5$ represents the orientation of the hand. The robot arm orientations are now determined through the various values in the domain of f. Thus the explicit form of the function f is a powerful mathematical model for the robot arm motioning. For instance, if we need to move the end effector in a circular helical manner, then the equation of the locus on the configuration space is

$$R(\theta) = (a\cos\theta, a\sin\theta, b\theta), \quad -\infty < \theta < \infty,$$

where a is the radius of the helix and $2\pi|b|$ is the pitch of the helix. So, by taking $l_3 + l_4 = a$ and $\theta_1 + \theta_2 + \theta_4 + \theta_5 = b\theta_1$, we can get the points in the joint space whose image is a helix in the configuration space by the function rule

$$f(\theta_1, \theta_2, \theta_4, \theta_5, l_4, l_3) = (a\cos\theta_1, a\sin\theta_1, b\theta_1).$$

This is just a sample for a function formula we can establish as per the desired motion of the hand. In general we can derive the clear mathematical expression for the robot end effector movements like the one given above. Thus the robot arm orientations are now determined through the various parameters in the domain of f. The explicit formula of f shall play definitely a powerful role in any robot movement algorithms.

We can now introduce variables $c_1 = \cos\theta_1$ and $s_1 = \sin\theta_1$ to simplify the expression given in (1). We will continue to use the co ordinate l_3 to specify the length of the segment 3. We will introduce another variable $t_1 = \theta_1 + \theta_2 + \theta_4 + \theta_5$ to represent the orientation of the hand. Then we have the polynomial representation as:

$$f(c_1, l_5, s_1, t_1) = \begin{bmatrix} (l_3 + l_4)c_1 \\ (l_3 + l_4)s_1 \\ t_1 \end{bmatrix} \qquad (2)$$

The main advantage of the formulation of such a polynomial map is that we have the complete definition of the configuration space and we can find easily where exactly the end effector lies according to the given values of the parameters in the joint space.

As the single workspace based representation is not enough to get the desired set of actions, the configuration space given in the figure 2 is very important. We can represent an action locating the end effector by using the values of the parameters in the domain of f. Like formula (1), the formula (2) can also be used as a model in any algorithms related to robot arm motioning of this type. The readers, who are interested in other technique like Groebner basis approach, can refer [G+09].

3. Graphical analysis and topological equivalence of the configuration space

The relations between topology and robotics run deep. In many ways, there has been a strong interplay between the two fields. Many robotic concepts have been motivated by the study of topological applications and many such topological applications have been best understood through topological concepts. The most basic topological construction in robotics is the configuration space which has already appeared in the earlier sections. In studying a system we often need to keep track of a set of variables associated with the position and arrangement of a collection of objects of interest. Note that the configuration space is a topological space that enables us to accomplish this variable tracking.

This section is almost independent of the past literature in the field of robotics. In this section we look at the configuration space of the robot arm we considered and we show how the topological space and construction that we have with us arise naturally in robotics. In fact, this new theory we developed, may conceive more new techniques and algorithms on robot motioning in future.

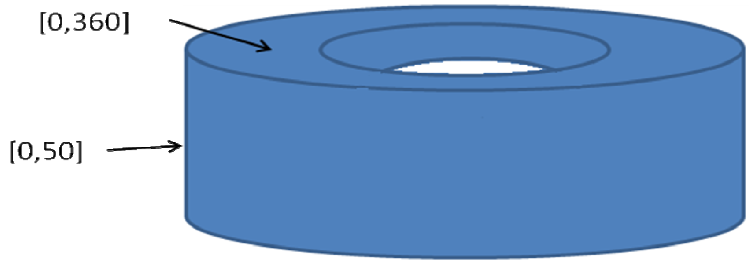

Figure 2. The configuration space for the robot arm mentioned in the section 3

In the base of the robot, arm can turn a complete rotation of 360 degrees. The corresponding configuration coordinate is a point on the circle S. The arm can turn from 0 to 360 degree and therefore the corresponding configuration space coordinate is contained to lie in the interval [0, 360]. We assume that the segment l_4 can extend from 0 to 50 centimeters, so its configuration space coordinate is constrained to lie in the interval [0, 50]. Thus the configuration space of this robot arm is S × [0, 360] × [0, 50], which is nothing but U× V as mentioned earlier. This is topologically equivalent to S ×I × I, a cylindrical shell as shown in the figure 2. Some more interesting topological problems on robotics can be found in [Bak90].

Now we define operational level space for the linkage which is the level surface of the space traced out by the end of the hand S_6. We can treat the linkage as a drawing device and the operational level space as the set of all points we can color with a pen at the end of the segment S_6. The resulting operational level space in this case is an annulus.

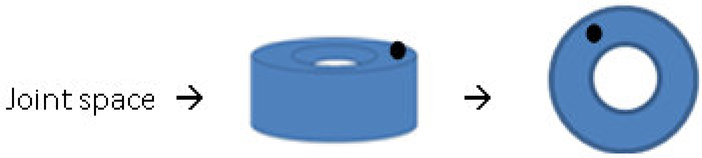

Figure 3. Composition of mappings from the joint space onto the operational level spaces

To each point in the configuration space of a mechanism, we associate the corresponding point in an operational level space. Actually, the function is the forward kinematic representation for the mechanism of the robot arm given above. Note that the map is continuous. This is because of two close points in the configuration space being compared to two close points in an operational level space.

From the topological point of view, configuration space and operational level spaces are especially interesting because it turned out that for any smooth closed manifold M, there exists a planar mechanical linkage such that one of the aspects of the configuration space of the latter is diffeomorphic to M. [KM02]

In the case of the robot arm considered above, f is a continuous map from the cylindrical shell into the family of annuli. The one particular case is shown in figure 3. An important question in the field of robotics is whether a given path in the operational level space of a mechanism can be traced by the hand. In our example, we can ask if there is a way to manipulate the configuration space variables θ_1, θ_2, θ_4, θ_5 and l_4 to yield the given path in an operational level space annulus. In other words, can the linkage draw a given line or curve or polygonal lines in the annulus? Definitely yes. Let us justify our answer with an example.

If z = g(x,y) is a point on the cylindrical shell, then g(x,y) = m represents a point on some operational level space of z = g(x,y) at z = m. So the end effector point on the configuration space can be located through the point (x, y) on the level surface g(x,y) = m. Thus we have a mapping correspondence between the joint space and operational level spaces. As this function is continuous and image of a path connected set is again path connected, a desired path in the configuration space can be obtained by a planar curve on the operational level space by properly defining the required path on it. For example, if we require to move the end effector in a circular path at an indicated height, say m, then it suffices to regulate the locus of the corresponding curve on the operational level surface i.e. also a circle, $x^2 + y^2 = k^2$ with z = m.

Thus, this type of functional correspondences is quite useful in robot motion planning. The above example is just an illustration and the mapping to be defined is motion and problem specific. By reducing the domain of the configuration space by 1 in operational space, the complexity of motion planning problems are also lessened. Therefore, it is promising for the inclusion of this concept of operational level space in future developments on robot arm motions.

Conclusion

In this paper, one complete method of algebraic as well as topological way of modeling a forward kinematic problem is introduced and discussed. The concept of the operational level surface is very new to the world of robots and future research work is encouraged to make use of this. The method can be applied to any general robot with little modifications in choosing the coordinate systems. This is actually one of the important methods which establish an equivalent system which preserves the major topological as well as algebraic properties. The rationale behind the selected algebraic method is that it always succeeds in solving parallel robot systems in all tested instances. This method for forward kinematic problem solving, can be implemented in several manipulator design aid tools such as posture analysis, equivalent mechanism validation, comparison between theoretical and real configurations, and singularity identification etc. Robot scientists are hereby advised to incorporate this modeling in their algorithms. Researchers can also proceed with the extension of the technique of modeling to robotic arms with joints having multiple arms as well as any new combination of arms.

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