Parameters Calculation Algorithm for Shock Waves of a Normal Laval Nozzle

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ABSTRACT: The article presents the fundamental equations (1), (2), (3), normal shock wave theory and numerical solving, if it is considered known quantities, in upstream v_1, p_1, ρ_1 of the shock wave. To obtain v_2, p_2, ρ_2 in a first stage the speed is computed (9). It examines the influence of speed (v_1) and pressure (p_1) of downstream on shock wave parameters. For $P0s = 6 \cdot 10^5$ $[N/m^2]$, $T0s = 500$ $[K]$ and $\delta = 2^0, 3^0, 4^0, 5^0$ angles, the pressure variations are calculated on the Laval nozzle. Are given positioning of the shock wave in the divergent nozzle.
KEYWORDS: Fluid Mechanics, Dynamics of gas, Turbo-jet

Introduction

A shock wave is a disturbance that propagates through a solid, liquid, gas, and in some cases by vacuum through a field such as electromagnetic and transport energy. Shock waves are characterized by a sudden, almost discontinuous environmental characteristics. Under certain conditions, in the supersonic nozzle divergent a normal shock wave can be stabilized. The shock wave is normal to the direction of movement. The fluid passing stationary shock wave parameters suddenly change their status.

Shock wave theory was founded by E. Carafoli, C. James, V.N. Constantinescu, L. Prandtl, L. Landau, P. Bradeanu.

In this article presents an algorithm for calculating the parameters of the downstream normal shock wave v_2, p_2, ρ_2, depending on the upstream parameters of the shock wave. The methodology presented allows positioning of the shock wave in the divergent nozzle and calculation of shock wave intensity.

1. Fundamental system of equations

It is considered a shock wave in the Laval nozzle axial symmetric in a divergence part. Let the index "1" wave upstream movement parameters and "2" for downstream parameters of the shock wave. The movement is considered non-stationary. The whole system gives a movement with a speed equal to the shock wave, but of opposite sign, so the move becomes permanent and shock wave is called stationary. Thus we can use the full form of conservation equations. They have the form:

- continuity equation

$$\rho_1 \cdot v_1 = \rho_2 \cdot v_2 \tag{1}$$

- momentum conservation equation

$$p_1 + \rho_1 \cdot v_1^2 = p_2 + \rho_2 \cdot v_2^2 \tag{2}$$

- energy conservation equation

$$\frac{\gamma}{\gamma-1} \cdot \frac{p_1}{\rho_1} + \frac{v_1^2}{2} = \frac{\gamma}{\gamma-1} \cdot \frac{p_2}{\rho_2} + \frac{v_2^2}{2} = \frac{\gamma \cdot R \cdot T_0}{\gamma-1}, \tag{3}$$

or

$$\frac{a^2}{\gamma-1} + \frac{v^2}{2} = \frac{\gamma+1}{2(\gamma-1)} a_c^2 = \frac{a_0^2}{\gamma-1} = \frac{\gamma R T_0}{\gamma-1} \tag{4}$$

plus the relations:

$$M = \frac{v}{a} = \frac{v}{\sqrt{\gamma R T}}, \quad M_1 = \frac{v_1}{a_1} = \frac{v_1}{\sqrt{\gamma \cdot R \cdot T_1}}, \quad M_2 = \frac{v_2}{a_2} = \frac{v_2}{\sqrt{\gamma \cdot R \cdot T2}}. \tag{5}$$

1.1. Solving numerical equations of the fundamental system

For the numerical solution of system (1), (2), (3), the unknown quantities v_2, p_2, ρ_2 depending on v_1, p_1, ρ_1 are adopted:

- adiabatic index $\gamma = 1,4$;
- air constant $R = 287,011 \left(\frac{J}{kg \cdot K}\right)$;
- Parameters upstream of the shock wave
  1. pressure: $p_1 = 1,5 \cdot 10^5 \left(\frac{N}{m^2}\right)$;
  2. density $\rho_1 = 1,23 \left(\frac{kg}{m^3}\right)$;
  3. velocity $v_1 = 450 \left(\frac{m}{s}\right)$.

The equations (1), (2), (3) becomes:

$$\rho_2 \cdot v_2 = \rho_1 \cdot v_1 = 553,5, \quad (6)$$

$$p_2 + \rho_2 \cdot v_2^2 = p_1 + \rho_1 \cdot v_1^2 = 3,991 \cdot 10^5, \quad (7)$$

$$\frac{\gamma}{\gamma - 1} \frac{p_2}{\rho_2} + \frac{v_2^2}{2} = \frac{\gamma}{\gamma - 1} \frac{p_1}{\rho_1} + \frac{v_1^2}{2} = 5,281 \cdot 10^5. \quad (8)$$

Remove the variables ρ_2 p_2 from the system (6), (7), (8) the equation resulting in v_2:

$$\frac{\gamma}{\gamma - 1} \cdot \frac{p_1 + \rho_1 \cdot v_1^2 - \rho_1 \cdot v_1 \cdot v2}{\frac{\rho_1 \cdot v_1}{v_2}} + \frac{v_2^2}{2} - 5,281 \cdot 10^5 = 0 \quad (9)$$

After the speed v_2 is calculated:

$$\rho_2(v_2) = \frac{\rho_1 \cdot v_1}{v_2}, \quad (10)$$

$$p_2(v_2) = p_1 + \rho_1 \cdot v_1^2 - \rho_1 \cdot v_1 \cdot v_2. \quad (11)$$

1.2. Numerical algorithm for solving the equation (9)

1. Enter function:

$$f(v_2) = \frac{\gamma}{\gamma - 1} \cdot \frac{p_1 + \rho_1 \cdot v_1^2 - \rho_1 \cdot v_1 \cdot v2}{\dfrac{\rho_1 \cdot v_1}{v_2}} + \frac{v_2^2}{2} - 5,281 \cdot 10^5,$$

2. Choose an approximate value for v_2:

To calculate the solution v_2 it used function "root"

$sol = root(f(v_2), v_2)$

For initial data adopted, we obtain the values:

$$sol = v_2 = 391,288 \ \left(\frac{m}{s}\right), \quad p_2 = 1,285 \cdot 10^5 \left(\frac{N}{m^2}\right), \quad \rho_2 = 1,415 \ \left(\frac{kg}{m^3}\right).$$

2. The influence of pressure p_1 on the downstream parameters of the shock wave

Using the following parameters:

$$\gamma = 1.4, \quad R=287.011, \quad v_1 = 500 \ [m/s], \quad \rho_1 = 1.23 \ \left[kg/m^3\right],$$

$$\zeta = 0..6$$

$$p_{1\zeta} = 1.5 \cdot 10^5 + \frac{\zeta}{20} \cdot 1.5 \cdot 10^5, \quad p_{2\zeta} = p_{1\zeta} + \rho_1 \cdot v_1^2 - \rho_2 \cdot v_2^2,$$

$$T_{1\zeta} = \frac{p_{1\zeta}}{R \cdot \rho_1}, \quad M_{1\zeta} = \frac{v_1}{\sqrt{\gamma \cdot R \cdot T_{1\zeta}}}.$$

and the function $F(v_2, \zeta)$:

$$F(v_2,\zeta)=\frac{\gamma}{\gamma-1}\cdot\frac{p_{1\zeta}+\rho_1\cdot v_1^2-\rho_1\cdot v_1\cdot v_2}{\frac{\rho_1\cdot v_1}{v_2}}+\frac{v_2^2}{2}-\left(\frac{\gamma}{\gamma-1}\cdot\frac{p_{1\zeta}}{\rho_1}+\frac{v_1^2}{2}\right)$$

we can peek an approximate value for v_2, and using the function "**root**"

$$v_2=400\quad[m/s]:$$

$$sol_\zeta=root(F(v_2,\zeta),v_2),\quad v_2=sol,$$

$$\rho_{2\zeta}=\frac{\rho_1\cdot v_1}{sol_\zeta},\quad p_{2\zeta}=p_{1\zeta}+\rho_1\cdot v_1^2-\rho_{2\zeta}\cdot(sol_\zeta)^2,$$

$$T_{2\zeta}=\frac{p_{2\zeta}}{R\cdot\rho_{2\zeta}},\quad M_{2\zeta}=\frac{sol_\zeta}{\sqrt{\gamma\cdot R\cdot T_{2\zeta}}}$$

Resulting numerical values for v_2, ρ_2, p_2, M_1, M_2

ζ	$v_{2\zeta}$ $[m/s]$	$\rho_{2\zeta}$ $[kg/m^3]$	$p_{2\zeta}$ $[N/m^2]$	$M_{1\zeta}$	$M_{2\zeta}$
0	367.886	1.359	$2.736\cdot10^5$	1.21	0.693
1	382.114	1.309	$2.739\cdot10^5$	1.181	0.706
2	396.341	1.262	$2.743\cdot10^5$	1.154	0.718
3	410.569	1.218	$2.747\cdot10^5$	1.128	0.731
4	424.797	1.177	$2.751\cdot10^5$	1.105	0.743
5	439.024	1.139	$2.755\cdot10^5$	1.082	0.754
6	453.252	1.103	$2.759\cdot10^5$	1.061	0.768

Graphic representation:

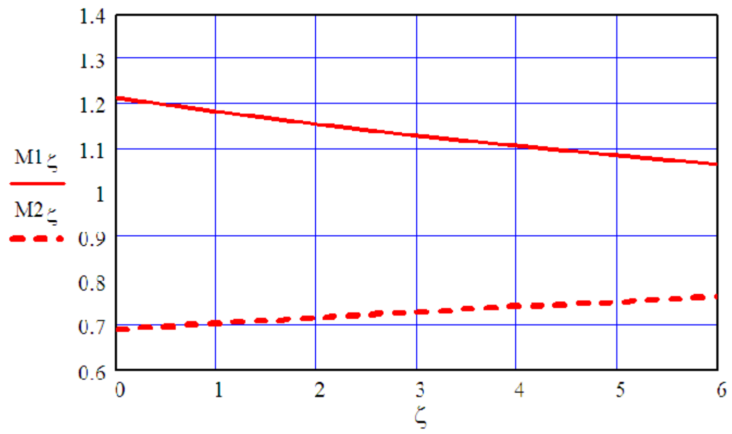

Figure 1. Mach number variations

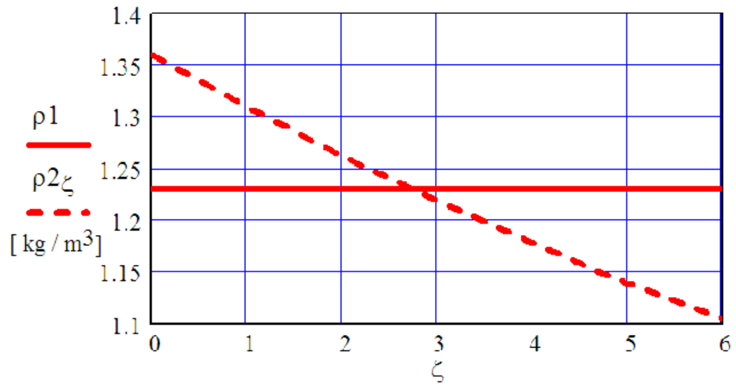

Figure 2. Density variation (ρ_2)

3. Influence of v_1 speed on the downstream of the shock wave parameters

It adopted the following initial data:

$$\gamma = 1.4, \quad R{=}287.011, \quad \rho_1 = 1.23 \ \left[kg/m^3\right], \quad p_1 = 2 \cdot 10^5 \ \left[N/m^2\right]$$

$$\zeta = 0..6, \quad v_{1\zeta} = 400 + 50 \cdot \zeta \ \ [m/s], \quad T_1 = \frac{p_1}{R \cdot \rho_1}, \quad T_1 = 566.534 \ \ [K]$$

The speed v_{2_ζ} is determined from the equation:

$$\left(v_{2\xi}\right)^2-\left(\frac{p_1}{\rho_1\cdot v_{1\xi}}+v_{1\xi}+\frac{\gamma-1}{\gamma}\cdot\rho_1\cdot v_{1\xi}\right)\cdot v_{2\xi}+\frac{\gamma-1}{2\cdot\gamma}\cdot\left(v_{1\xi}\right)^2=0$$

It result the value:

ξ	$M_{1\xi}$	$M_{2\xi}$	$T_{2\xi}$ [K]	$p_{2\xi}$ $[N/m^2]$
0	1.048	0.531	654.115	$3.402\cdot10^5$
1	1.153	0.509	681.768	$3.918\cdot10^5$
2	1.258	0.487	711.914	$4.505\cdot10^5$
3	1.362	0.467	744.384	$5.156\cdot10^5$
4	1.467	0.448	779.124	$5.868\cdot10^5$
5	1.572	0.431	818.125	$6.639\cdot10^5$
6	1.677	0.417	855.402	$7.489\cdot10^5$

4. Divergent part of nozzle influence $(2\cdot\delta^o)$, on pressure variation

It examines four types of flaring of the divergent part:

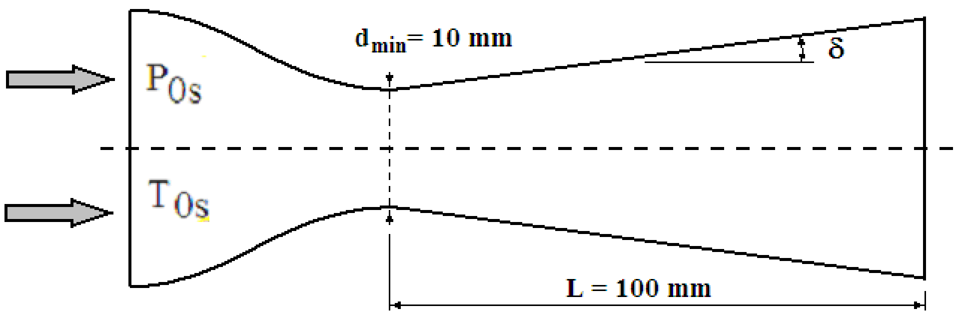

Figure 3. Laval nozzle

We adopted:

$$P0s = 6 \cdot 10^5 \left[\frac{N}{m^2}\right], \quad T0s = 500 \quad [K]$$

Case 1 $(w = 1), \quad \delta_1 = 2^o$.

$$S(L,w) := \frac{\pi}{4} \cdot \left(dmin + 2 \cdot L \cdot \tan(\delta_w)\right)^2$$

It follows

$$Dm_{max} = S_{min} \cdot \sqrt{P_{0s} \cdot \rho_{0s}} \cdot \psi_{cr}, \quad \psi_{cr} = \sqrt{\frac{2 \cdot \gamma}{\gamma + 1} \cdot \frac{P_{0s}}{\rho_{0s}}}$$

$$Qm_{max} = Dm_{max} = 0.085 \quad [kg/s]$$

and variation of pressure along the divergent nozzle:

L $[m]$	$\delta_1 = 2^o$	$\delta_2 = 3^o$	$\delta_3 = 4^o$	$\delta_4 = 5^o$
	P(L,1) $[N/m^2]$	P(L,2) $[N/m^2]$	P(L,3) $[N/m^2]$	P(L,4) $[N/m^2]$
0.01	$1.754 \cdot 10^5$	$1.493 \cdot 10^5$	$1.295 \cdot 10^5$	$1.137 \cdot 10^5$
0.02	$1.296 \cdot 10^5$	$1.010 \cdot 10^5$	$8.113 \cdot 10^4$	$6.657 \cdot 10^4$
0.03	$1.010 \cdot 10^5$	$7.340 \cdot 10^4$	$5.629 \cdot 10^5$	$5.727 \cdot 10^5$
0.04	$8.121 \cdot 10^4$	$5.629 \cdot 10^5$	$5.751 \cdot 10^5$	$5.826 \cdot 10^5$
0.05	$6.670 \cdot 10^4$	$5.726 \cdot 10^5$	$5.826 \cdot 10^5$	$5.884 \cdot 10^5$
0.06	$5.628 \cdot 10^5$	$5.793 \cdot 10^5$	$5.874 \cdot 10^5$	$5.919 \cdot 10^5$
0.07	$5.697 \cdot 10^5$	$5.840 \cdot 10^5$	$5.907 \cdot 10^5$	$5.942 \cdot 10^5$
0.08	$5.751 \cdot 10^5$	$5.874 \cdot 10^5$	$5.929 \cdot 10^5$	$5.957 \cdot 10^5$
0.09	$5.792 \cdot 10^5$	$5.900 \cdot 10^5$	$5.945 \cdot 10^5$	$5.968 \cdot 10^5$
0.10	$5.826 \cdot 10^5$	$5.919 \cdot 10^5$	$5.957 \cdot 10^5$	$5.975 \cdot 10^5$

Conclusions

The fundamental system of equations (1), (2), (3) is considered known the quantities p_1, ρ_1, v_1, that depend on parameters of stagnation P_0, T_0, and nozzle geometry.

Calculation algorithm allows the positioning of the shock wave and determine its intensity $(p_{av,UŞ} / p_{amUŞ})$. The scheme of Fig. 3 for data $P(L,1)$ with $\delta_1 = 2^0$, we plot (fig.4):

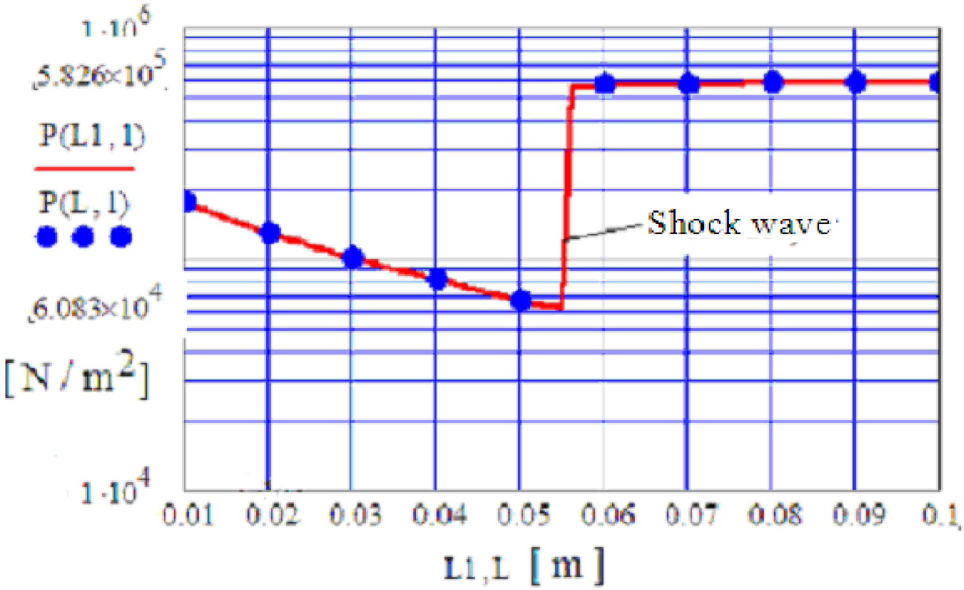

Fig. 4. Variation of pressure along the divergent Laval nozzle, with $\delta_1 = 2^0$

The resulting emergence of the shock wave at a distance $x_1 = 55$ mm of critical section, with an intensity $(p_{av,UŞ} / p_{amUŞ})_1 = \frac{5,826 \cdot 10^5}{6,083 \cdot 10^4} = 9,578$.

For $P(L,4)$, and $\delta_4 = 5^0$ result the variation (fig.5)

Shock wave:

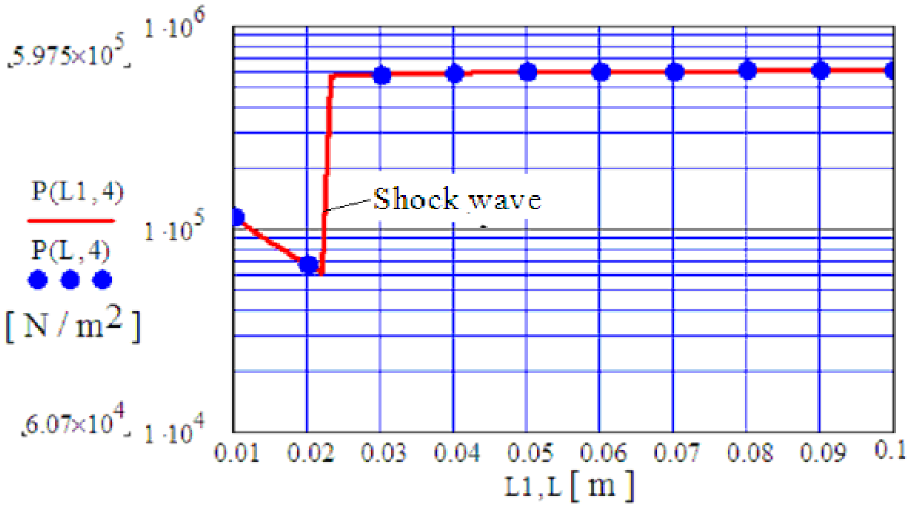

Fig. 5. Variation of pressure in the divergent Laval nozzle, $\delta_4 = 5^{\circ}$

The resulting emergence of the shock wave at a distance $x_1 = 22$ mm of critical section, with an intensity $\left(p_{av,UŞ} / p_{amUŞ}\right)_4 = \frac{5{,}975 \cdot 10^5}{6{,}070 \cdot 10^4} = 9{,}843$.

Positioning of the shock wave in the Laval nozzle divergent length allows selecting a part of the diverging shock wave does not occur.

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