COMPARATIVE STUDY ON SOME METHODS OF HANDLING NONLINEAR EQUATIONS

A.R. Tasiu, A. Abbas, M.N. Alhassan and A.N. Umar

Department of Mathematics, Usmanu Danfodiyo University, Sokoto, Nigeria

Corresponding author: Rufai Tasiu Ahmad, ahmadtgk02@gmail.com

ABSTRACT: *The study of numerical solutions for nonlinear equations is one of the trending areas of research in pure and applied mathematics. As it is applied to several areas such as optimization, mathematical modelling, operation research and so on. This research work compared three methods (Bijection, Secant and Newton-Raphson) for solving nonlinear equations. It was observed that secant may converge faster than Bisection method, while Newton-Raphson method converge faster than secant method. Therefore, we concluded that Newton-Raphson method is the most efficient and fastest method for solving nonlinear equation than secant method and Bijection method because it converges at the three alterations.*
KEYWORDS: *Nonlinear, Bijection, Secant, Newton-Raphson, convergent*

1. INTRODUCTION

Numerical analysis is the area of mathematics and computer science that creates, analyzes, and implements algorithms for solving numerically the problems of continuous mathematics. Such problems originate generally from real-world applications of algebra, geometry, and calculus, and they involve variables which vary continuously [8]. These problems occur throughout the natural sciences, social sciences, medicine, engineering, and business. Beginning in the 1940's, the growth in power and availability of digital computers has led to an increasing use of realistic mathematical models in science, medicine, engineering, and business; and numerical analysis of increasing sophistication has been needed to solve these more accurate and complex mathematical models of the world.The formal academic area of numerical analysis varies from highly theoretical mathematical studies to computer science issues involving the effects of computer hardware and software on the implementation of specific algorithms [2].
A rough categorization of the principal areas of numerical analysis is given below, keeping in mind that there is often a great deal of overlap between the listed areas. In addition, the numerical solution of many mathematical problems involves some combination of some of these areas, possibly all of them. There are also a few problems which do not fit neatly into any of the following categories [2].
Root finding problem is a problem of finding a root of the equation f(x) = 0, where f(x) is a function of a single variable. Let $f(x)$ be a function, we are interested in finding x = ε such that $f(\varepsilon) = 0$. The number ε is called the root or zero of $f(x)$. $Ff(x)$ may be algebraic, trigonometric or transcendental function Ehiwario and Aghamie [7].
In this journal we will compare Bisection, Secant, and Newton-Raphson method for solving non-linear equations.

Some Definitions

Bisection Method: Given a function $f(x) = 0$, continuous on a closed interval $[a, b]$, such that $f(a)f(b) < 0$, then, the function $f(x) = 0$ has at least a root or zero in the interval $[a, b]$. The method calls for a repeated halving of subintervals of $[a, b]$ containing the root. The root always converges, though it is very slow in converging.

Newton-Raphson Method: Given a function $f(x) = 0$, chooses where xk as a trial value for the root at the nth step and the approximate value of the next step $k + 1$ using

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} \qquad (1)$$

Secant Method: Given a function f(x) = 0, continuous on a closed interval [a, b], such that f(a)f(b) < 0 , then, the function f(x) = 0 has at least a root or zero in the interval [a, b]. The method uses the formula:

$$xk + 1 = xk - \frac{f(x_k)(x_k - x_{k-1})}{f(x_k) - f(x_{k-1})} \qquad (2)$$

2. METHODOLOGY

This section deals with the methodology and algorithms for Bisection, Newton-Raphson and Secant method, some examples were solved, and also the results of the methods were compared.

Bisection Method

Given a function $f(x) = 0$, continuous on a closed interval $[a; b]$, such that $f(a)f(b) < 0$, then, the function $f(x) = 0$ has at least a root or zero in the interval $[a; b]$. The method calls for a repeated halving of subintervals of $[a, b]$ containing the root. The root always converges, though it is very slow in converging.

Algorithm of Bisection Method

Input:

(i) $f(x)$ that is the given function

(ii) a_0, b_0 are the two numbers, such that $f(a_0)f(b_0)<0$.

Output: (i) An approximation of the root of f(x) = 0 in $[a_0, b_0]$, for k =0, 1, 2, 3,….. do until satisfied.

Compute

$$ck = \frac{ak+bk}{2} \qquad (3)$$

- Test if c_k is the desired root. If so stop.
- If c_k is not the desired root, test if $f(ck)f(ak) < 0$. If so, set $bk + 1 = c_k$ and $ak + 1 = c_k$ otherwise set $c_k = bk + 1 = bk$

End.

Stopping Criteria for Bisection Method

The following are the stopping criteria as suggested by Biswa (2002) Let ε be the error tolerance, that is we would like to obtain the root with an error of at most of ε Then, accept k = ck as a root of f(x) = 0: If any of the following criteria is satisfied:

(i) $\left|f(ck)\right| \leq \varepsilon$

(ii) $\dfrac{|c_{k-1} - ck|}{|c_{k|}|} \leq \varepsilon$

(iii) $(\dfrac{b-1}{2^k}) \leq \varepsilon$

The number of iterations k is greater than or equal to a predetermined number, say N:

Theorem 1.1.1. The number of iterations, N needed in the Bisection method to obtain an accuracy of ε is given by:

$$N \geq \frac{\log_{10}^{(b_0 - a_0)} - \log_{10}^{(\varepsilon)}}{\log_{10}^{2}} \qquad (4)$$

Remark 1.1.1. Since the number of iterations, N needed to achieve a certain accuracy depends upon the initial length of the interval containing the root, it is desirable to choose the initial interval [a0, b0] as small as possible**.**

Problem 1.1.1.: Obtain the positive root of $x^3 - 6x^2 + 11x - 6 = 0$ using bisection method.

Solution. Find the interval $[a, b]$ bracketing the root:

Since the bisection method find a root in a given interval $[a, b]$, we try to find the interval first. This can be done using IVT:

Choose $a_0 = 2.5$ and $b_0 = 4$

We now show that both hypotheses of IVT are satisfies for $f(x)$ in $[2.5, 4]$.

(i) $x^3 - 6x^2 + 11x - 6 = 0$is continuous on $[2.5, 4]$

(ii) $f(2.5)f(4) < 0$

Thus, by IVT there is a root of $f(x) = 0$ in $[2.5, 4]$.

Input Data:

- ≤ 10.5507
- $a_0 = 2.5$, $b_0 = 4$

First iteration $(k = 0)$

$c0 = 3.25$

Since

$f(c_0)f(a_0) = f(3.25)f(2.5) < 0, set\ b_1 = c_0, a_1 = a_0$.

Second iteration: $(k = 1)$

$c_1 = 2.8750$

Since $f(c_1)f(a_1) > 0$, set $a_2 = c_1, b_2 = b_1$.

Third iteration: $(k = 2)$

c_2=3.0625

Since $f(c_2)f(a_2) < 0, set\ b_3 = c_2, a_3 = a_2$.

Fourth iteration $(k = 3)$:

c_3=2.9688

Since f(c3)f(a3) > 0,set a4 = c3, b4 = b3.

Fifth iteration (k=4):

C_4=3.0157

Since f(c4)f(a4) < 0, set b5 = c2, a5 = a4.

Sixth iteration (k=5):

$c5 = 2.9923$

Since f(c5)f(a5) > 0, set a6 = c5, $b_6 = b_5$.

Seventh iteration (k=6):

$c6 = 3.004$

Since f(c6)f(a6) < 0, set b7 = c6, $a_7 = a_6$.

Eight iteration (k=7)

$c7 = 2.9982$

Since f(c7)f(a7) > 0, set a8 = c_7, $b_8 = b_7$.

Ninth iteration (k=8):

C_8=3.0011

Since f(c8)f(a8) < 0, set b9 = c8, a9 = a8.

Tenth iteration (k=9):

C_9=2.9977

Since f(c9)f(a9) > 0, set $a1_0$ = c9, b10 = b9.

Eleventh iteration (k=10):

C_{10}=3.0004

Let ε = 0.0005.

Therefore f(x) converges to 3. Since

$$\left|\frac{c10 - c9}{c9}\right| < 0.0005$$

Problem 1.1.2. Find the minimum number of iterations needed by the bisection algorithm to approximate the root

x = 3 or $x^3 - 6x^2 + 11x - 6 = 0$ with error tolerance 10^{-1}.

Solution. Input Data
(i) End points of the interval: a = 2.5, b = 4
(ii) Error tolerance: $\varepsilon = 10^{-3} = 0.001$
Using equation 4 we have
by substitution the values of (a0 and b0) in equation 4, we get

$$N\&\geq \frac{\log_{10}^{(1.5)} - \log_{10}^{(10^{-3})}}{\log_{10}^{2}} \quad (5)$$

≤ 10.5507
Thus the minimum of 11 iterations is needed to obtain the desired accuracy using the bisection method.

Newton-Raphson Method
The function f(x) = 0 can be expanded in the neighborhood of the root x0 through the Taylor polynomial expansion.
where x can be seen as a trial value for the root at the nth step and the approximate value of the next step x_{k+1} can be derived from

$$f(x_{k+1}) = f(x_k) + (x_{k+1} - x_k)f'(x_k) = 0 \quad (6)$$

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} k = 0,1,2 \quad (7)$$

called the Newton-Raphson Method.
Algorithm of the Newton-Raphson Method
Input:
(i) f(x) that is the given function
(ii) x_0 is the initial approximation
(iii) ε is the error tolerance
(iv) N : is the maximum number of iteration
Output: An approximation to the root x = ε or a message of a failure
Assumption: x = ε is a simple root of f(x) = 0.

- compute f(x) and $f'(x)$
- compute

$$xk + 1 = xk - \frac{f(xk)}{f'(xk)} \text{ k=0,1,2,3,…….}$$

Do until convergence or failure

- if $|f(xk)| < \varepsilon$,or $|\frac{|xk+1-xk|}{xk}| < \varepsilon$
- End ε

Remark 1.1.2 If none of the above criteria has been satisfied, within a predetermined, say, N iteration, then the method has failed after the prescribed number of iterations. In this case, one could try the method again with a different xo. Meanwhile, a judicious choice of xo can sometimes be obtained by drawing the graph of f(x), if possible. However, there does not seems to exist a clear- cut guideline on how to choose a right starting point, xo that guarantees the convergence of the Newton-Raphson method to a desire root.

Problem 1.1.3. Obtain the positive root of $x^3 - 6x^2 + 11x - 6 = 0$ using Newton-Raphson method.
Solution
Obtain the initial approximation:
Since from problem 1.1.2 the positive root must lie between 2.5 and 4. So let's take $x_0 = 3.25$
Input Data:
(i) The function f(x) = $x^3 - 6x^2 + 11x - 6 = 0$
(ii) Initial approximation x_0 = 3.25
Using Equation 1.5,
Compute $f'(x) = 3x^2 - 12x + 11$
First Iteration (k=0)
X_1=3.0593
Second Iteration (k=1)
X_2=3.0046
Third Iteration (k=2)
X_3=3
Fourth Iteration (k=3)
X_4=3
Hence f(x) converges to 3 since $|x_{k+1} - x_k| = 0$

The Secant Method
Given a function f(x) knowing $f(x_k - 1)$ we can approximate $f'(x)$ as

$$f'(x) = \frac{f(x)-f(xk-1)}{x_k - x_{k-1}} \quad (8)$$

substituting equation (8) into equation (7), we have

$$xk + 1 = xk - \frac{f(x_k)(x_k - x_{k-1})}{f(x_k) - f(x_{k-1})} \quad (9)$$

equation (9) is referred to as Secant Iteration Method.
Algorithm of the Secant Method
Input:

i. f(x) that is the given function
ii. x0, x1 that is the two initial approximation of the root ε
iii. ε that is the error tolerance, and
iv. N that is the maximum number of iterations

Output: An approximation of the exact solution ε or a message of failure, for
k = 1, 2,….. do until convergence or otherwise.

- Compute f(xk) and f(xk+1)
- Compute the next approximation using equation (1.9)
- Test of Iterations: If $|x_{k+1} - x_k| < \varepsilon$or k > N, stop.
- End

Problem 1.1.4. Find the positive root of $f(x) = x^3 - 6x^2 + 11x - 6 = 0$ using Secant method.
Solution
Obtaining an initial approximation. We use the initial values 2.5 and 4 as in Problem 1.1.2, i.e x_0 = 2.5, x1 = 4
Input Data:
(i) the function $f(x) = x^3 - 6x^2 + 11x - 6 = 0$
(ii) Initial approximations x0 = 2.5, x1 = 4

Using equation (1.7)
First iteration (k=1)
X_2=2.5882
Second iteration (k=2)
X_3=2.6733
Third iteration (k=3)
X_4=4.5575
Fourth iteration (k=4)
X_5=2.7648
Fifth iteration (k=5)
X_6=2.7648
Sixth iteration (k=6)
X_7=3.3486
Seventh iteration (k=7)
X_8=2.9161
Eighth iteration (k=8)
X_9=2.9740
Ninth iteration (k=9)
X_{10}=3.0038
Tenth iteration (k=10)
X_{11}=3

3. RESULTS

Comparing the results of the three methods under investigation, we observed that the rates of convergence of the methods are in the following order: Newton Raphson method, Secant method, and Bisection method. This is in line with the findings of [10]. Comparing the Bisection method and the Secant method, we noticed that Secant may converge faster than Bisection method. However, Also comparing Secant method and Newton's method, Newton's method requires the evaluation of both the function f(x) and its derivative at every iteration while Secant method only requires the evaluation of f(x). Hence, Secant method may occasionally be faster. In Allen and Isaacson (1998), it was argued that if we assume that evaluating f(x) takes as much time as evaluating its derivative, and we neglect all other costs, we can do two iterations of Secant (decreasing the logarithm of error by factor $\alpha^2 = 2.6$) for the same cost as one iteration of Newton-Raphson method (decreasing the logarithm of error by a factor 2).

CONCLUSION

In this paper, algorithm for finding the root of nonlinear equations using bisection method, secant method and Newton-Raphson method were given, and also nonlinear equations were solved using these methods. Comparing the results of the methods it was discovered that Secant will converge faster than Bisection method, while Newton-Raphson method converge faster than secant method. Therefore Newton-Raphson method is the most efficient and fastest method for solving nonlinear equation.

REFERENCES

[1] **Amat S. Busquier, S. Gutierrez J.M.** (2003). Geometric constructions of Iterative functions to Solve Nonlinear Equations, Journal of Computer. Appl. Math.157, 197-205.

[2] **Atkinson K. E.** (2007). An Introduction to Numerical Analysis, 2nd ed., John Wiley & Sons, New.

[3] **Autar K.K. and Egwu E.** (2008), http://www.numericalmethods.eng.usf.edu

[4] **Charles A. and W.W. Cooper** (2008), Non-linear Power of Adjacent Extreme Points Methods in Linear Programming, Econometrica Volome. 25(1) pp132-153.

[5] **Datta B. N.** (2012), Lecture Notes on Numerical Solution of root Finding Problems. www.math.niu.edu/dattab.

[6] **Ehiwario J.C**. (2013), Lecture Notes on MTH213: Introduction to Numerical Analysis, College of Education Agbor. An A_liate of Delta State University, Abraka.

[7] **Ehiwario J.C., Aghamie, S.O.** (2014). Comparative Study of Bisection, Newton Raphson and Secant Methods of Root- Finding Problems. IOSR Journal of Engineering, 4 (4), 01-07.

[8] **Iwetan C.N, Fuwape I.A., Olajide M.S. and Adenodi R.A.** (2012), Comparative Study of the Bisection and Newton Methods in solving for Zero and Extremes of a Single-Variable Function. J. of NAMP vol.21 pp 173-176.

[9] **McDonough J.M.** (2001), Lecture in Computational Numerical Analysis. Department of Mechanical Engineering, University of Kentucky.

[10] **Srivastava R.B. and Srivastava S.** (2011), Comparison of Numerical Rate of Convergence of Bisection, Newton and Secant Methods. Journal of Chemical,Biological and Physical Sciences. Vol 2(1) pp 472-479.

[11] **Stoer J. Bulrisch R.** (1993). Introduction to Numerical Analysis,Second edition, Springer - Verlag.