THE NEW ITERATIVE METHOD FOR SOLVING LINEAR AND NONLINEAR SYSTEMS OF PARTIAL DIFFERENTIAL EQUATIONS

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ABSTRACT: *In this paper, we used the New Iterative Method (NIM) developed by Daftardar-Gejji and Jafari for the solution of linear and nonlinear systems of partial differential equations. This method is very simple as it reduces the size of computation and readily converges to the exact solution. To demonstrate the efficiency of the method, some illustrative examples were provided. The results obtained confirmed that the method is an efficient method for a wide variety of systems of linear and nonlinear PDEs.*

KEYWORDS: *New Iterative Method, Systems, Partial Differential Equations, Linear, Nonlinear*

1. INTRODUCTION

Systems of partial differential equations, linear or nonlinear, have attracted much concern in studying evolution equations that describe wave propagation, in investigating shallow water waves, and in examining the chemical reaction-diffusion model of Brusselator [1]. Recently, many powerful methods have been presented in solving the partial differential equation systems, such as Variational Iterative Method (VIM) [2-3], Adomian Decomposition Method (ADM) [4-5], Aboodh Adomian Decomposition Method (AADM) [6], Laplace Adomian Decomposition Method (LADM) [7], Homotopy Perturbation Method [8], Differential Transformation Method (DTM) [9].

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In this paper, we applied the New Iterative Method (NIM) to obtain the solution of partial differential equation systems. This method was developed by Daftardar-Gejji and Jafari in 2006 [10] and has been extensively used by many researchers for the treatment of linear and nonlinear ordinary and partial differential equations of integer and fractional order such as fractional physical differential equations [11], higher order KDV [12], Fractional Gas Dynamics and Coupled Burger's Equations [13], Nonlinear Abel type integral equation[14], linear and nonlinear Klein-Gordon Equation [15] and many more.

This method is highly accurate and requires reduced amount of calculations compared with the existing iterative methods.

Considering a nonlinear system of partial differential equation in an operator form as

$$\begin{aligned} L_t u + R_1(u,v,w) + N_1(u,v,w) &= g_1 \\ L_t v + R_2(u,v,w) + N_2(u,v,w) &= g_2 \\ L_t w + R_3(u,v,w) + N_3(u,v,w) &= g_3 \end{aligned} \quad (1)$$

With initial data

$$u(x,y,o) = f_1(x)$$
$$v(x,y,o) = f_2(x)$$
$$w(x,y,o) = f_3(x)$$

Where L_t is considered a first order partial differential operator, R_j , $1 \le$ j $\le$ 3 and N_j , $1 \le$ j $\le$ 3 , are linear and nonlinear operators respectively, and g_1, g_2 and g_3 are source terms [Wazwaz]

2. NEW ITERATIVE METHOD (NIM)

To illustrate the idea of the NIM, we consider the following general functional equation:

$$u = f + N(u) \quad (2)$$

where N is a nonlinear operator and f is a given function. We can find the solution of equation (2) having the series form

$$u = \sum_{i=0}^{\infty} u_i \quad (3)$$

The nonlinear operator N can be decomposed as:

$$N\left(\sum_{i=0}^{\infty} u_i\right) = N(u_0) + \sum_{i=1}^{\infty}\left\{N\left(\sum_{j=0}^{i} u_j\right) - N\left(\sum_{j=0}^{i-1} u_j\right)\right\} \quad (4)$$

Substituting equations (3) and (4) into equation (2) gives

$$\sum_{i=0}^{\infty} u_i = f + N(u_0) + \sum_{i=1}^{\infty}\left\{N\left(\sum_{j=0}^{i} u_j\right) - N\left(\sum_{j=0}^{i-1} u_j\right)\right\} \quad (5)$$

We define the recurrence relation of equation in the following way:

$$\begin{aligned} u_0 &= f \\ u_1 &= N(u_0) \\ u_2 &= N(u_0 + u_1) - N(u_0) \\ u_3 &= N(u_0 + u_1 + u_2) - N(u_0 + u_1) \\ u_{n+1} &= N(u_0 + u_1 + \dots + u_n) - N(u_0 + u_1 + \dots + u_{n-1}); \quad n = 1, 2, 3 \end{aligned} \quad (6)$$

then

$$u_1 + \cdots + u_{m+1} = N(u_0 + u_1 + \cdots + u_m);$$

$m = 1,2,3$

and

$$\sum_{i=0}^{\infty} u_i = f + N\left(\sum_{j=0}^{\infty} u_j\right) \quad (7)$$

The approximate solution of (2) is given by

$$u(x,t) = u_0 + u_1 + u_2 + u_3 + \ldots \quad (8)$$

3. APPLICATIONS

In this section, we used the New Iterative Method (NIM) to solve homogeneous and inhomogeneous linear nonlinear system of partial differential equations.

3.1: The homogeneous linear system of PDE

Consider the homogeneous linear system

$$u_t + v_x = 0$$
$$v_t + u_x = 0 \quad (9)$$

IC: $u(x,0) = e^x$, $v(x,0) = e^{-x}$

The exact solutions are:

$$u(x,t) = e^x Cosh\, t + e^{-x} Sinh\, t$$
$$v(x,t) = e^{-x} Cosh\, t - e^x Sinh\, t$$

according to the New Iterative Method (NIM), we have

$$u(x,t) = f_1 + \int_0^t (-v_x)dt \quad (10)$$
$$v(x,t) = f_2 + \int_0^t (-u_x)dt \quad (11)$$
$$N(u_k) = \int_0^t (-v_{kx})dt$$
$$N(v_k) = \int_0^t (-u_{kx})dt \quad (12)$$

from the initial condition, we have

$$f_1 = u_0 = e^x \ and\ f_2 = v_0 = e^{-x}$$

using (12), When k = 0,

$$N(u_0) = u_1 = \int_0^t (-v_{0x})dt$$
$$= \int_0^t -\,-e^{-x}dt = \int_0^t e^{-x}dt = te^{-x}$$
$$u_1 = te^{-x}$$
$$N(v_0) = v_1 = \int_0^t (-v_{0x})dt$$
$$= \int_0^t -e^x dt = \int_0^t -e^x dt = -te^x$$
$$u_1 = te^{-x}$$
$$v_1 = -te^x$$
$$u_2 = N(u_0 + u_1) - N(u_0)$$
$$u_2 = \frac{t^2}{2} e^x$$
$$v_2 = N(v_0 + v_1) - N(v_0)$$
$$v_2 = \frac{t^2}{2} e^{-x}$$

Accordingly, we obtain the successive approximations:

$$u_0 = e^x,\ v_0 = e^{-x}$$
$$u_1 = te^{-x},\quad v_1 = -te^x \quad (13)$$
$$u_2 = \frac{t^2}{2!} e^x,\ v_2 = \frac{t^2}{2!} e^{-x}$$
$$u_3 = \frac{t^2}{3!} e^{-x},\ v_3 = -\frac{t^2}{3!} e^x$$

Therefore the solution is as follows

$$u(x,t) = \sum_{n=1}^{\infty} u_n = e^x + te^{-x} + \frac{t^2}{2!} e^x + \frac{t^2}{3!} e^{-x} + \cdots$$
$$v(x,t) = \sum_{n=1}^{\infty} v_n = e^{-x} - te^x + \frac{t^2}{2!} e^{-x} - \frac{t^2}{3!} e^x + \cdots$$
$$u(x,t) = e^x\left(1 + \frac{t^2}{2!} + \frac{t^4}{4!} + \cdots\right) + e^{-x}\left(t + \frac{t^2}{3!} + \frac{t^5}{5!} + \cdots\right)$$
$$v(x,t) = e^{-x}\left(1 + \frac{t^2}{2!} + \frac{t^4}{4!} + \cdots\right) - e^x\left(1 + \frac{t^2}{3!} + \frac{t^5}{5!} + \cdots\right)$$

We know that the Taylor series of:

$$Cosh\, t = 1 + \frac{t^2}{2!} + \frac{t^4}{4!} + \cdots \quad and$$
$$Sinh\, t = 1 + \frac{t^2}{3!} + \frac{t^5}{5!} + \cdots$$

Therefore, the exact solutions are:

$$u(x,t) = e^x Cosh\, t + e^{-x} Sinh\, t$$
$$v(x,t) = e^{-x} Cosh\, t - e^x Sinh\, t \quad (14)$$

3.2: The inhomogeneous linear system

Consider the inhomogeneous linear system

$$u_t - v_x - (u - v) = -2$$
$$v_t + u_x - (u - v) = -2 \quad (15)$$

IC: $u(x,0) = 1 + e^x$, $v(x,0) = -1 + e^{-x}$

The exact solutions are:

$$u(x,t) = 1 + e^{x+t}$$
$$v(x,t) = -1 + e^{x-t}$$

according to the New Iterative Method (NIM), we have

$$u(x,t) = f_1 + \int_0^t \big(-2 + v_x + (u - v)\big)dt \quad (16)$$
$$v(x,t) = f_2 + \int_0^t \big(-2 - u_x + (u - v)\big)dt \quad (17)$$
$$N(u_k) = \int_0^t \big(-2 + v_{kx} + (u_k - v_k)\big)dt$$
$$N(v_k) = \int_0^t \big(-2 - u_{kx} + (u_k - v_k)\big)dt \quad (18)$$

from the initial condition, we have

$$f_1 = u_0 = 1 + e^x \ and\ f_2 = v_0 = -1 + e^{-x}$$

using (18), When k = 0,

$$N(u_0) = u_1 = \int_0^t \big(-2 + v_{0x} + (u_0 - v_0)\big)dt$$
$$= \int_0^t -2 + e^x + 2dt = \int_0^t e^x dt = te^x$$

$$u_1 = te^x$$
$$N(v_0) = v_1 = \int_0^t (-2 - u_{0x} + (u_0 - v_0))dt = \int_0^t -e^x dt = -te^x$$
$$v_1 = -te^x$$
$$u_2 = N(u_0 + u_1) - N(u_0)$$
$$u_2 = \frac{t^2}{2!}e^x$$
$$v_2 = N(v_0 + v_1) - N(v_0)$$
$$v_2 = \frac{t^2}{2!}e^x$$

Accordingly, we obtain the successive approximations:

$$\begin{aligned} u_0 &= 1 + e^x \\ v_0 &= -1 + e^x \\ u_1 &= te^x \\ v_1 &= -te^x \\ u_2 &= \frac{t^2}{2!}e^x \\ v_2 &= \frac{t^2}{2!}e^x \\ u_3 &= \frac{t^2}{3!}e^x \\ v_3 &= -\frac{t^2}{3!}e^x \end{aligned} \tag{19}$$

Therefore the solution is as follows

$$u(x,t) = \sum_{n=1}^{\infty} u_n = 1 + e^x\left(1 + t + \frac{t^2}{2!} + \frac{t^2}{3!} + \cdots\right)$$
$$v(x,t) = \sum_{n=1}^{\infty} v_n = -1 + e^x\left(1 - t + \frac{t^2}{2!} - \frac{t^2}{3!} + \cdots\right)$$

We know that the Taylor series of:

$$e^t = 1 + t + \frac{t^2}{2!} + \frac{t^2}{3!} + \cdots \quad and \quad e^{-t} = 1 - t + \frac{t^2}{2!} - \frac{t^2}{3!} + \cdots$$

Therefore, the exact solutions are:

$$\begin{aligned} u(x,t) &= 1 + e^x.e^t = 1 + e^{x+t} \\ v(x,t) &= -1 + e^x.e^{-t} = 1 + e^{x-t} \end{aligned} \tag{20}$$

3.3: The inhomogeneous nonlinear system

Consider the inhomogeneous nonlinear system

$$\begin{aligned} u_t + vu_x + u &= 1 \\ v_t - uv_x - v &= 1 \end{aligned} \tag{21}$$

IC: $u(x,0) = e^x$, $v(x,0) = e^{-x}$

The exact solutions are:

$$u(x,t) = e^{x-t}$$
$$v(x,t) = e^{-x+t}$$

according to the New Iterative Method (NIM), we have

$$u(x,t) = f_1 + \int_0^t (1 - vu_x - u)dt$$
$$v(x,t) = f_2 + \int_0^t (1 + uv_x + v)dt$$
$$N(u_k) = \int_0^t (1 - v_k u_{kx} - u_k)dt \quad \text{and}$$
$$N(v_k) = \int_0^t (1 + u_k v_{kx} + v_k)dt \tag{22}$$

from the initial condition, we have

$$f_1 = u_0 = e^x \ \ and \ \ f_2 = v_0 = e^{-x}$$

using (22), When k = 0,

$$N(u_0) = u_1 = \int_0^t (1 - v_0 u_{0x} - u_0)dt$$
$$= \int_0^t 1 - 1 - e^x dt = \int_0^t -e^x dt = -te^x$$
$$u_1 = -te^x$$
$$N(v_0) = v_1 = \int_0^t (1 + u_0 v_{0x} + v_0)dt = \int_0^t e^{-x} dt = te^{-x}$$
$$v_1 = te^{-x}$$
$$u_2 = N(u_0 + u_1) - N(u_0)$$
$$u_2 = \frac{t^2}{2!}e^x$$
$$v_2 = N(v_0 + v_1) - N(v_0)$$
$$v_2 = \frac{t^2}{2!}e^{-x}$$

Accordingly, we obtain the successive approximations:

$$\begin{aligned} u_0 &= e^x, \\ v_0 &= e^{-x} \\ u_1 &= -te^x \\ v_1 &= te^{-x} \\ u_2 &= \frac{t^2}{2!}e^x \\ v_2 &= \frac{t^2}{2!}e^{-x} \\ u_3 &= -\frac{t^2}{3!}e^x \\ v_3 &= \frac{t^2}{3!}e^{-x} \end{aligned} \tag{23}$$

Therefore the solution is as follows

$$u(x,t) = \sum_{n=1}^{\infty} u_n = e^x\left(1 - t + \frac{t^2}{2!} - \frac{t^2}{3!} + \cdots\right)$$
$$v(x,t) = \sum_{n=1}^{\infty} v_n = e^{-x}\left(1 + t + \frac{t^2}{2!} + \frac{t^2}{3!} + \cdots\right)$$

We know that the Taylor series of:

$$e^{-t} = 1 - t + \frac{t^2}{2!} - \frac{t^2}{3!} + \cdots \quad and$$
$$e^{+t} = 1 + t + \frac{t^2}{2!} + \frac{t^2}{3!} + \cdots$$

Therefore, the exact solutions are:

$$\begin{aligned} u(x,t) &= e^x.e^{-t} = e^{x-t} \\ v(x,t) &= e^{-x}.e^t = e^{-x+t} \end{aligned} \tag{24}$$

3.4: The homogeneous nonlinear system

Consider the homogeneous nonlinear system

$$\begin{aligned} u_t + u_x v_x + u_y v_y + u &= 0 \\ v_t + v_x w_x - v_y w_y - v &= 0 \\ w_t + w_x u_x + w_y u_y - w &= 0 \end{aligned} \tag{25}$$

IC: $u(x,y,0) = e^{x+y}$, $v(x,y,0) = e^{x-y}$, $w(x,y,0) = e^{-x+y}$

The exact solutions are:

$$u(x,y,t) = e^{x+y-t}$$
$$v(x,y,t) = e^{x-y+t}$$
$$w(x,y,t) = e^{-x+y+t}$$

according to the New Iterative Method (NIM), we have

$$u(x,t) = f_1 + \int_0^t (-u_x v_x - u_y v_y - u)dt \qquad (26)$$
$$v(x,t) = f_2 + \int_0^t (v_y w_y + v - v_x w_x)dt \qquad (27)$$
$$v(x,t) = f_3 + \int_0^t (w - w_x u_x - w_y u_y)dt \qquad (28)$$

Therefore,

$$N(u_k) = \int_0^t (-u_{kx} v_{kx} - u_{ky} v_{ky} - u_k)dt$$
$$N(v_k) = \int_0^t (v_{ky} w_{ky} + v_k - v_{kx} w_{kx})dt \qquad (29)$$
$$N(w_k) = \int_0^t (w_k - w_{kx} u_{kx} - w_{ky} u_{ky})dt$$

from the initial condition, we have

$f_1 = u_0 = e^{x+y}$, $\quad f_2 = v_0 = e^{x-y}$
and $f_3 = w_0 = e^{-x+y}$

using (29), When k = 0,

$$N(u_0) = u_1 = \int_0^t (-u_{0x} v_{0x} - u_{0y} v_{0y} - u_0)dt = \int_0^t -e^{x+y} dt = -te^{x+y}$$
$$u_1 = -te^{x+y}$$
$$N(v_0) = v_1 = \int_0^t (v_{0y} w_{0y} + v_0 - v_{0x} w_{0x})dt = \int_0^t e^{x-y} dt$$
$$v_1 = te^{x-y}$$
$$N(w_0) = w_1 = \int_0^t (w_0 - w_{0x} u_{0x} - w_{0y} u_{0y})dt = \int_0^t e^{-x+y} dt$$
$$w_1 = te^{-x+y}$$
$$u_2 = N(u_0 + u_1) - N(u_0) = \frac{t^2}{2!} e^{x+y}$$
$$v_2 = N(v_0 + v_1) - N(v_0) = \frac{t^2}{2!} e^{x-y}$$
$$w_2 = N(w_0 + w_1) - N(w_0) = \frac{t^2}{2!} e^{-x+y}$$

accordingly, we obtain the successive approximations:

$$u_0 = e^{x+y}$$
$$v_0 = e^{x-y}$$
$$w_0 = e^{-x+y}$$
$$u_1 = -te^{x+y}$$
$$v_1 = te^{x-y}$$
$$w_1 = te^{-x+y}$$
$$u_2 = \frac{t^2}{2!} e^{x+y}, \quad v_2 = \frac{t^2}{2!} e^{x-y} \qquad (30)$$
$$w_2 = \frac{t^2}{2!} e^{-x+y}$$
$$u_3 = -\frac{t^2}{3!} e^{x+t}$$
$$v_3 = \frac{t^2}{3!} e^{x-y}$$
$$w_3 = \frac{t^2}{3!} e^{-x+y}$$

Therefore the solution is as follows

$$u(x,y,t) = \sum_{n=1}^{\infty} u_n = e^{x+y}\left(1 - t + \frac{t^2}{2!} - \frac{t^2}{3!} + \cdots\right)$$
$$v(x,y,t) = \sum_{n=1}^{\infty} v_n = e^{x-y}\left(1 + t + \frac{t^2}{2!} + \frac{t^2}{3!} + \cdots\right)$$
$$w(x,y,t) = \sum_{n=1}^{\infty} v_n = e^{-x+y}\left(1 + t + \frac{t^2}{2!} + \frac{t^2}{3!} + \cdots\right)$$

The Taylor series of:

$$e^{-t} = 1 - t + \frac{t^2}{2!} - \frac{t^2}{3!} + \cdots \quad and$$
$$e^{+t} = 1 + t + \frac{t^2}{2!} + \frac{t^2}{3!} + \cdots$$

therefore, the exact solutions are:

$$u(x,y,t) = e^{x+y}.e^{-t} = e^{x+y-t}$$
$$v(x,y,t) = e^{x-y}.e^{t} = e^{x-y+t} \qquad (31)$$
$$w(x,y,t) = e^{-x+y}.e^{t} = e^{-x+y+t}$$

CONCLUSION

In this work, the New Iterative Method (NIM) is employed for the solution of systems of linear and nonlinear partial differential equations. This method was applied to four examples successfully and the results show that NIM is a powerful and efficient Mathematical tool for solving systems of linear and nonlinear partial differential equations. The method reduces the size of computational work thus the method is powerful and effective and can be utilized to tackle complex situations arising out of real world.

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