SOME APPLICATION OF EIGENVALUE AND EIGENVECTOR PROBLEMS

Zahidullah Rehan

**Sayed Jamaluddin Afghani University, Education Faculty,
Mathematics and physics Department, Kunar, Afghanistan**

Corresponding author: Zahidullah Rehan rehanzahidullah@gmail.com

ABSTRACT: One of the most useful brand of mathematics is linear algebra, which have more application in science and engineering, because it required the description of some measurable quantities. In this research paper we determine some application of Eigen-value problems. So in this research work first we discuss how to obtain the Eigen-value and Eigen-vector of square matrix and their characteristic equation and polynomial and then apply the solution of Eigen-value problem to stretching of elastic membrane problems, eigenvalue problems arising from population model and vibrating system of two masses on two springs problems.
KEYWORDS: Matrix, Eigenvalue, Eigenvector

1. INTRODUCTION

From the view point of engineering application eigenvalue and eigenvector problem are more useful in the connection with matrices. Linear equation $Ax = b$ come from steady state problem. [8] Eigenvalue have their great importance in dynamic problems the solution of $\frac{dx}{dt} = Ax$ is changing with time growing or decaying or oscillating we cannot find if by elimination. Here we discuss linear algebra based on $Ax = \lambda x$. all the matrices in this paper or square matrices. [1]
Let A be $n \times n$ matrix then consider the following vector equation. [4]

$$Ax = \lambda x \tag{1}$$

In the above equation x and λ are uknown where x is vector and λ is scalar, we must determine the value of these unknown which satisfier equation (1).
We know that $x = 0$ is the solution of equation (1) for the value of λ, because $A0 = 0$. This is of no interest. The solution of equation (1) for the value of λ is known as an Eigen-value (characteristic value) of the matrix A and the solution of equation (1) are known as the Eigen-vector (characteristic vector) of A. Corresponding to that Eigen value of A is called the Spectrum of A. [6]
On the base of eigenvalue and eigenvector and eigenmode characterization of reducible matrix does not have necessary eigenvector. If the reducible matrix has Eigen-value the Eigen-value is not necessarily unique and has a finite value. Eigen-vector of the corresponding to the Eigen-value of the reducible matrix is not unique and contain at least a finite element. The regular reducible matrix does not have a unique Eigen-mode will all element are finite. [2]
According to our information and advance technology the complexity of the social communication and biological networks surrounding us is increasing rapidly. Many important combinatorial parameters of large networks are often hard to calculate are approximate. Eigen-values provide an effective and efficient tools for studying properties of large graphs which arise in practice. [4]

2. HOW TO FIND THE EIGEN-VALUE AND EIGEN-VECTOR OF A MATRIX

The problem of deterring the eigenvalue and eigenvector of a matrix is called an Eigen-value problem. To find the Eigen-value and Eigen-vector of a matrix first we write equation (1) in the component. [6]

$$\left.\begin{array}{c} a_{11}x_1 + \cdots\cdots\cdots + a_{1n} = \lambda x_1 \\ a_{21}x_1 + \cdots\cdots\cdots + a_{2n} = \lambda x_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{n1}x_1 + \cdots\cdots\cdots + a_{nn} = \lambda x_n \end{array}\right\} \tag{2}$$

Transform the term on the right side to the left side, we have

$$\left.\begin{array}{c} (a_{11} - \lambda)x_1 + a_{12}x_2 + \cdots\cdots\cdots + a_{1n} = 0 \\ a_{21}x_1 + (a_{22} - \lambda)x_2 + \cdots\cdots\cdots + a_{2n} = 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots\cdots\cdots + (a_{nn} - \lambda) = \lambda x_n \end{array}\right\} \tag{3}$$

In matrix notation equation (3) can be written as

$$(A - \lambda I)x = 0 \tag{4}$$

By Cramer theorem the above homogenous linear system has a non-trivial solution if and only if the determinant of the coefficient is zero.

$$\det(A-\lambda I)=\begin{vmatrix} a_{11}-\lambda & a_{12} & \cdots\cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots\cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots\cdots & a_{nn}-\lambda \end{vmatrix}=0 \qquad (5)$$

Here $(A-\lambda I)$ is known as characteristic matrix and $p(\lambda)$ is the characteristic determinant of A. Equation (5) is known as the characteristic equation of A, by developing $p(\lambda)$ we obtain a polynomial of nth degree in λ. this is known as the characteristic polynomial of A.

Theorem.
If w and x are eigenvector of a matrix A corresponding to the same eigenvalue λ, os are $w+x$ (provided $x\neq -w$) and kx for any $k\neq 0$.
Hence the Eigen-vectors corresponding to one and the same Eigen-value λ of A, together with 0, form a victor space called the eigenvalue space of A corresponding to that λ. [7]

Proof.
$Aw=\lambda w$ And $Ax=\lambda x$ imply

$$A(w+x)=Aw+Ax=\lambda w+\lambda x=\lambda(w+x)$$

And

$$A(kw+=k((Aw)=k(\lambda w)=\lambda(kw)$$

Hence

$$A(kw+lx)=\lambda(kw+lx).$$

In particular, an Eigen-vector x is determined only up to a constant factor. Hence we can normalize$x$, that is, multiply it by a scalar to get a unite vector. The following problems will illustrate that a $n\times n$ matrix may have n linearly independent eigenvector or it may have fewer than n.

Problem: determine the eigenvalue and eigenvector of $A=\begin{pmatrix}7 & 3\\ 3 & -1\end{pmatrix}$. [9]

Solution: the characteristic polynomial is

$$\det(A-\lambda I)=\begin{vmatrix}7-\lambda & 3\\ 3 & -1-\lambda\end{vmatrix}$$
$$=(7-\lambda)(-1-\lambda)-(3)(3)$$
$$=\lambda^2-6\lambda-16$$

So the characteristic equation is $\lambda^2-2\lambda+2=0$ from this equation we obtain the velue of λ as

$$\lambda^2-6\lambda-16=(\lambda-8)(\lambda+2)=0$$
$$\lambda=8\,,\lambda=-2$$

So that the required eigenvalues are $\lambda=8\,,\lambda=-2$.
Now putting the value of λ in the matrix $\begin{bmatrix}7-\lambda & 3\\ 3 & -1-\lambda\end{bmatrix}$ we obtain.

$$\begin{bmatrix}7-8 & 3\\ 3 & -1-8\end{bmatrix}=\begin{bmatrix}-1 & 3\\ 3 & 9\end{bmatrix}$$

Now we solve the equation $B\bar{x}=\bar{0}$,

$$\begin{bmatrix}-1 & 3\\ 3 & -9\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$
$$\left[\begin{array}{cc|c}-1 & 3 & 0\\ 3 & -9 & 0\end{array}\right]\rightarrow\left[\begin{array}{cc|c}-1 & 3 & 0\\ 0 & 0 & 0\end{array}\right]$$
$$-x_1+3x_2=0\Rightarrow 3x_2=x_1$$
$$x_1=1\ ,\quad x_2=3$$

Hence the eigenvector is $\begin{bmatrix}3\\ 1\end{bmatrix}$. And eigenvalue is 8.
Now we obtain the eigenvector for $\lambda=-2$. Now putting the value of λ in the matrix $\begin{bmatrix}7-\lambda & 3\\ 3 & -1-\lambda\end{bmatrix}$ we obtain.

$$\begin{bmatrix}7+2 & 3\\ 3 & -1+2\end{bmatrix}=\begin{bmatrix}9 & 3\\ 3 & 1\end{bmatrix}$$

Now we solve the equation $B\bar{x}=\bar{0}$,

$$\begin{bmatrix}9 & 3\\ 3 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$
$$\left[\begin{array}{cc|c}9 & 3 & 0\\ 3 & 1 & 0\end{array}\right]\rightarrow\left[\begin{array}{cc|c}0 & 0 & 0\\ 3 & 1 & 0\end{array}\right]$$
$$3x_1+x_2=0\Rightarrow x_2=-3x_1$$
$$x_1=1\ ,\quad x_2=-3$$

Hence the eigenvector is $\begin{bmatrix}1\\ -3\end{bmatrix}$. And eigenvalue is -2.

Problem: determine the eigenvalue and eigenvector of $A=\begin{pmatrix}8 & -8 & -2\\ 4 & -3 & -2\\ 3 & -4 & 1\end{pmatrix}$. [2]
Solution: the characteristic polynomial is

$$\det(A-\lambda I)=\begin{vmatrix}8-\lambda & -8 & -2\\ 4 & -3-\lambda & -2\\ 3 & -4 & 1-\lambda\end{vmatrix}$$

$$\lambda^3-[sum\ of\ digonal\ elemnts]\lambda^2+[sum\ digonal\ minors]-|A|=0$$
$$\lambda^3-[8-3+1]\lambda^2+[-11+14+8]-6=0$$
$$\lambda^3-6\lambda^2+11-6=0$$

So the characteristic equation is $\lambda^2-2\lambda+2=0$ from this equation we obtain the velue of λ as

$$\lambda=1,2,3$$

So that the required eigenvalues are $\lambda = 1, 2, 3$. Now putting the value of λ in the matrix $\begin{bmatrix} 8-\lambda & -8 & -2 \\ 4 & -3-\lambda & -2 \\ 3 & -4 & 1-\lambda \end{bmatrix}$ we obtain

$$\begin{bmatrix} 8-1 & -8 & -2 \\ 4 & -3-1 & -2 \\ 3 & -4 & 1-1 \end{bmatrix} = \begin{bmatrix} 7 & -8 & -2 \\ 4 & -4 & -2 \\ 3 & -4 & 0 \end{bmatrix}$$

Now we solve the equation $B\bar{x} = \bar{0}$,

$$\begin{bmatrix} 7 & -8 & -2 \\ 4 & -4 & -2 \\ 3 & -4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now we solve the above system by Crammer's rule

$$\left.\begin{aligned} 7x_1 - 8x_2 - 2x_3 = 0 \\ 4x_1 - 4x_2 - 2x_3 = 0 \end{aligned}\right\}$$

$$\frac{x_1}{\begin{vmatrix} -8 & -2 \\ -4 & -2 \end{vmatrix}} = \frac{x_2}{\begin{vmatrix} 7 & -2 \\ 4 & -2 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 7 & -8 \\ 4 & -4 \end{vmatrix}}$$

$$\frac{x_1}{8} = \frac{x_2}{6} = \frac{x_3}{4}$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 8 \\ 6 \\ 4 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix}$$

Hence the eigenvector is $\begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix}$. And eigenvalue is 1.

Similarly we can find the eigenvectors for $\lambda = 2,3$.

3. APPLICATION OF EIGENVALUE AND EIGENVECTORS

Eigenvalues problems have greatest application in daily life. Some of the application which belonge to engineering, pahysics and mathematice we stydied here.

3.1. Expansion of Elastic Membrane

Let us consider an elastic membrane in with boundary circle $x_1^2 + x_2^2 = 1$ is expand from the point $P(x_1, x_2)$ to $Q(y_1, y_2)$ which given by

$$y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = Ax = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad (6)$$

Obtain the principal direction of the position vector x of P for which the f the position vector y of Q have the similar or exactly opposite direction, find the shape of the circle after the instability? [6]

Solution. We are looking for a vector x such that $y = \lambda x$. Since $y = Ax$, this gives $Ax = \lambda x$, the equation of an eigenvalue problem. In component, $Ax = \lambda x$ is

$$\left.\begin{aligned} (5-\lambda)x_1 + 3x_2 = 0 \\ 3x_1 + (5-\lambda)x_2 = 0 \end{aligned}\right\} \qquad (7)$$

The characteristic equation is

$$\begin{vmatrix} 5-\lambda & 3 \\ 3 & 5-\lambda \end{vmatrix} = (5-\lambda)^2 - 9 = 0$$

$$\lambda = 8, 2$$

For $\lambda = 8$ system (7) becomes

$$\left.\begin{aligned} -3x_1 + 3x_2 = 0 \\ 3x_1 + 3x_2 = 0 \end{aligned}\right|$$

$$solution\ x_2 = x_1, x_1\ arbitrary. for\ instance, x_1 = x_2 = 1$$

For $\lambda = 2$ system (7) becomes

$$\left.\begin{aligned} 3x_1 + 3x_2 = 0 \\ 3x_1 - 3x_2 = 0 \end{aligned}\right|$$

$$solution\ x_2 = -x_1, x_1\ arbitrary. for\ instance, x_1 = 1, x_2 = -1$$

We thus obtain as eigenvector of A, for instance, $[1 \quad 1]^T$ corresponding to λ_1 and $[1 - 1]^T$ corresponding to λ_2.

One of the vectors make 45^o degree angle and the other vector make 135^o degree angle with the positive direction of x_1. So we say that the eigenvalues in this problem shows that the principal direction of the given membrane is expand by factors 8 and 2 respectively.

Now let us consider the principal direction as direction of a new Cartesian coordinate system, in which positive u_1-semi-axis in the 1st quadrant and the positive u_2- semi-axis in the 2nd quadrant of the old system, let suppose that $u_1 = r\cos\theta, u_2 = r\sin\varphi$, then the boundary before the expansion of the circular membrane has coordinates $\cos\varphi, \sin\varphi$ and after the expansion the coordinate is

$$z_1 = 8\cos\varphi \qquad z_2 = 2\sin\varphi$$

Since $\cos^2\varphi + \sin^2\varphi = 1$, which show that the shape of an elastic membrane is ellipse.

$$\frac{z_1^2}{64} + \frac{z_2^2}{4} = 1$$

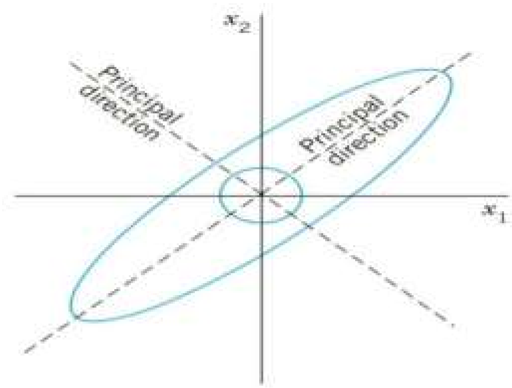

Figure 1. Undeformed and deformed membrane in Example 1

3.2. Eigenvalue Problems Arising from Population Model

The Leslie model describes age-specified population growth, as follows. Let age attained by the females in some animal population be 9 years. Divide the population into three age classes of 3 years each. Let the "Leslie matrix" be

$$l = [l_{jk}] = \begin{bmatrix} 0 & 2.3 & 0.4 \\ 0.6 & 0 & 0 \\ 0 & 0.3 & 0 \end{bmatrix} \qquad (8)$$

Where l_{1k} is the average numbers of daughter borne to a single female during the time she is in age class k, and $l_{j,j-1}$ $(j = 2,3)$ is the fraction of females in the age class $j - 1$ that will survive and pass into class j?

a) Find the number of females in each class after 3,6,9 years if each class initially consists of 400 females?

b) In which initial distribution the number of females in each class change by the same proportion and also find the rate of change? [3]

Solution:

(a) initially $x_0^T = [400 \quad 400 \quad 400]$, after three years,

$$x_{(3)} = Lx_{(0)} = \begin{bmatrix} 0 & 2.3 & 0.4 \\ 0.6 & 0 & 0 \\ 0 & 0.3 & 0 \end{bmatrix} \begin{bmatrix} 400 \\ 400 \\ 400 \end{bmatrix} = \begin{bmatrix} 1080 \\ 240 \\ 120 \end{bmatrix}$$

On the same way the number of females after six years is $x_{(6)}^T = \left(Lx_{(3)}\right)^T = [600 \quad 648 \quad 72]$, and after nine years this value equal to

$$x_{(9)}^T = \left(Lx_{(6)}\right)^T = [1519.2 \quad 360 \quad 195.4]$$

(b) proportional change means that we are looking for a distribution vector x such that $Lx = \lambda x$, where λ is the rate of change
(growth if $\lambda > 1$, decrease if $\lambda < 1$).
The characteristic equation is

$$\det(L - \lambda I) = -\lambda^3 = 0.6(-2.3\lambda - 0.3 \cdot 0.4)$$
$$= -\lambda^3 + 1.38\lambda + 0.072 = 0$$

A positive root is find to be (for instance, by Newton's method) $\lambda = 1.2$. A corresponding eigenvector x can be determined from the characteristic matrix.

$$A - 1.2I = \begin{bmatrix} -1.2 & 2.3 & 0.4 \\ 0.6 & -1.2 & 0 \\ 0 & 0.3 & -1.2 \end{bmatrix},$$

$$say, \quad x = \begin{bmatrix} 1 \\ 0.5 \\ 0.125 \end{bmatrix}$$

Where $x_3 = 0.125$ is selected, then we obtain $x_2 = 0.5$ from $0.3x_2 - 1.2x_3 = 0$, and $x_1 = 1$ from $-1.2x_1 + 2.3x_2 + 0.4x_3 = 0$. For the population we multiply x by $\frac{1200}{1+0.5+0.125} = 738$. Proportional growth of the numbers of females in the three classes will occur if the initial values are $738, 369, 92$ in classes $1, 2, 3$ respectively. The growth rate will be 1.2 per 3 years.

3.3. Vibrating System of Two Masses on Two Springs

Mas-spring system involving several masses and spring can be treated as eigenvalue problems. For instance, the mechanical system in Fig. 2 is governed by the system of ODEs

$$\left.\begin{aligned} y_1'' &= -5y_1 + 2y_2 \\ y_2'' &= 2y_1 - 2y_2 \end{aligned}\right\} \qquad (9)$$

Where y_1 and y_2 are the displacements of the masses from rest, as shown in the figure, and prime denote derivatives with respect to time. In vector form, this becomes [6]

$$y'' = \begin{bmatrix} y_1'' \\ y_2'' \end{bmatrix} = Ay = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \qquad (10)$$

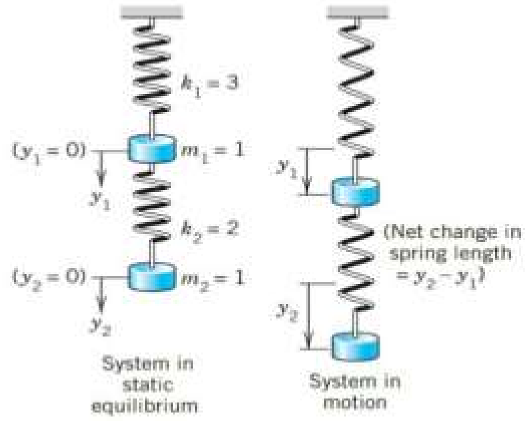

Fig. 2. Masses on spring in example 4

We try a vector solution of the form

$$y = e^{\omega t} \qquad (11)$$

This is suggested by a mechanical system of a single mass on a spring. Whose motion is given by exponential function (and sines and cosines). Substitution into (7) gives

$$\omega^2 x e^{\omega t} = Axe^{wt}$$

Dividing by e^{wt} and writing $\omega^2 = \lambda$, we see that our mechanical system leads to the eigenvalue problem

$$Ax = \lambda x \qquad (12)$$
$$where\, \lambda = \omega^2$$

Here the eigenvalues are $\lambda = -1, -6$ consequently $\omega = \pm I \text{ and } \pm i\sqrt{6}$.

Corresponding eigenvectors are

$$x = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \end{bmatrix} \tag{13}$$

From equation (12) we obtain the following complex solution.

$$x_1 e^{\pm it} = x_1(\cos t \pm \sin t)$$

$$x_2 e^{\pm i\sqrt{6}t} = x_2\left(\cos\sqrt{6}t \pm \sin\sqrt{6}t\right)$$

By addition and subtraction we get the following four real solutions.

$$x_1 \cos t, \quad x_1 \sin t, \quad x_2 \cos\sqrt{6}t, \quad x_2 \sin\sqrt{6}t$$

A general solution is obtained by taking a linear combination of these.

$$y = x_1(a_1 \cos t + b_1 \sin t) + x_2(a_2 \cos\sqrt{6}t + b_2 \sin\sqrt{6}t)$$

With arbitrary constants a_1, b_1, a_2, b_2 (to which value can be assigned by prescribing initial displacement and initial velocity of each of the two mass), the component of y are

$$y_1 = a_1 \cos t + b_1 \sin t + 2a_2 \cos\sqrt{6}t + 2b_2 \sin\sqrt{6}t$$

$$y = 2a_1 \cos t + 2b_1 \sin t - a_2 \cos\sqrt{6}t - b_2 \sin\sqrt{6}t$$

These function describe harmonic oscillation of the two masses. Physically, this hold to be expected because we have neglected damping.

4. CONCLUSIONS

Eigenvalue problem are of the greatest interest to the real life. Such that we have seen in that the application of the problem are useful in mathematics and physics for instance in analyzing problems involving Stretching of Elastic Membrane problems, Eigenvalue Problems Arising from Population Model problems and Vibrating System on Two Masses of Two Springs problems. This procedure also used for calculating some other physical and biological problem like arising of Markov processes problem. Beside the application of the eigenvalue and eigenvector problem we find the eigenvalue and eigenvector of the given square matrix.

Conflicts of interest

The author declare no conflict of interest regarding the publication of this paper.

REFERENCE

[1]. **Abreu Nair, Cardoso Domingos, França Francisca, Vinagre Cybele**. (2016). On main eigenvalues of certain graphs. 10.13140/ RG.2.1.2863.1926.

[2]. **Himmatul Mursyidah, Subiono Eigenvalue**, Eigenvector, eigenmode of reducible matrix and its application, AIP conference proceeding 1867, 020044 (2017), https://doi.org/10.1063/ 1.4994447

[3]. **Brouwer A, Haemers W**. (2005) Eigenvalues and perfect matchings. Lin Algebra Appl 395:155–162.

[4]. **Cioaba Sebastian.** (2011). Some Applications of Eigenvalues of Graphs. Structural Analysis of Complex Networks. 10.1007/978-0-8176-4789-6_14.

[5]. **Fernandes Rafael, Judice Joaquim, Trevisan Vilmar.** (2017). Complementary eigenvalues of graphs. Linear Algebra and its Applications. 527. 10.1016/j.laa. 2017.03.029.

[6]. **Keryszig, Erwin.** 2006. Advance engineering mathematics. 9th edition. Willy India.

[7]. **Li Jianxi, Chang An.** (2008). Some applications on the method of eigenvalue interlacing for graphs [J]. Journal of Mathematical Research & Exposition. 28. 10.3770/j.issn:1000-341X.2008.02.002.

[8]. **Prince Tanvir, Angulo Nieves.** (2014). Application of Eigenvalues and Eigenvectors and Diagonalization to Environmental Science. Applied Ecology and Environmental Sciences. 2. 106-109. 10.12691/aees-2-4-4.

[9]. **Ramane Harish Chandra, Revankar Deepak, Patil J.B.** (2013). Bounds for the degree sum eigenvalues and degree sum energy of a graph. International Journal of Pure and Applied Mathematical Sciences. 6. 161-167