CONSTRUCTION OF PAIRWISE BALANCED DESIGNS USING LOTTO DESIGN WHEN Λ =1 AND WHEN, Λ =2

[1] Popoola, O. P. and [2] Oyejola, B. A.

[1] Mathematics and Statistics Department, the Polytechnic, Ibadan Oyo State, Nigeria
[2] Department of Statistics, University of Ilorin, Ilorin Nigeria

ABSTRACT: Pairwise Balanced Designs (PBDs) are of fundamental importance in combinatorial theory, it has many applications in the constructions of other types of designs while Lotto Designs (LDs) are also combinatorial. Some scholars construct PBDs from Balanced Incomplete block Designs, Transversal Designs (TD), Truncated Transversal Designs, Projective Planes and Geometries. This research work tends to use **lotto designs to construct a pairwise balanced designs** when λ =1 and when, λ =2. The problem was divided into three stages; We identified PBDs that qualify as LDs. A FORTRAN program based on appropriate computer algorithm was written; some conditions was imposed on the LDs that was constructed so as to ensure their compliance with PBDs; construction of some specific LDs was done from the parents PBDs through the writing of another computer program that utilized the binary search to generating subsets of a specific universal set in the Microsoft office access database context. PBD(6,{2,3},1) produced LD(6,3,4,3), (6,3,5,3) and (6,3,6,3) when λ =1 and when, λ =2. PBD(6,{2,3},1) produced LDs(6,3,3,3), (6,3,4,3), and(6,3,6,3).

For each of the parents PBDs used to produced LDs we then test the LDs produced so as to ensure conformity to the underlying properties of PBDs, this was done by using the second software**.** Thus, PBDs(4,{3,4},1), (5,{3,10},1)and(6,{3,19},1) was constructed and PBDs(3,{3,1},1), (4,{3,4},1) and (6,{3,19},1) was constructed respectively from the LDs.

KEYWORDS: Pairwise Balanced Designs, Lotto Designs, Designs Theory and Combinatorial Theory

1. INTRODUCTION

One of the most important breakthroughs in design theory was made by Richard Wilson in the early 1970s. He showed that the trivial necessary conditions for the existence of various kinds of designs are asymptotically sufficient. For example, given a positive integer *k* >1, the necessary conditions for the existence of a 2(*v*,*k*,1) design are that *k*−1 divides *v*−1 and *k*(*k*−1) divides *v*(*v*−1); it follows from Wilson's theorem that, given *k*, a design exists for all but finitely many *v* satisfying these conditions. [Wil72]

Wilson ([Wil74]) Defined a pairwise balanced design (or PBD) as a design (V, B) such that every pair of distinct points is contained in exactly λ blocks, where λ is a positive integer.

For the purpose of describing Wilson's existence theory, we define a *pairwise balanced design*, or *PBD*, to be a set *V* of points with a collection *B* of subsets (called blocks) such that any two Points lie in a unique block. If *K* is the set of cardinalities of blocks, we refer to such a design as a PBD (*K*). Thus, if *K* = {*k*}, a PBD (*K*) with *v* points is a Steiner system *S* (2, *k, v*). We allow the possibility that 0, 1 ∈ *K*. Indeed, adding blocks with cardinality at most 1 does not affect the defining property of a PBD. Apart from this, a PBD is the same thing as a *linear space*.

Colbourn and Dinitz ([CD06]) defined a Pairwise Balanced Design (PBD) for a set K of positive integers. A design(v, K, λ) consisting of an ordered pair (V, B) where Va set of size in v and B is a collection of subsets of V with the property that every pair of elements of Voccurs in exactly λ blocks, and for every block$B\epsilon \mathrm{B}, |\mathrm{B}|\epsilon K$. When $\lambda = 1$, a pairwise balanced design is called a (v, K)-PBD.

Necessary Conditions for the existence of a (v, K)-PBD:

- $v \equiv 1(mod \propto (K))$
- $v(v-1) \equiv 0(mod \beta(K))$

where $\propto (K)$ is the greatest common divisor of the integers $\{k-1|k\epsilon K\}$ and $\beta(K)$ is the greatest common divisor of the integers$\{k(k-1)|k\epsilon K\}$.

Mullin and Gronau ([MG95]) defined a Pairwise designs as Let K be a subset of positive integers and let λ be a positive integer. A pairwise balanced design (PBD (v, K, λ) or (K, λ)-PBD) of order v with block sizes from K is a pair (V, B), where V is a finite set (the point set) of cardinality v and B is a family of subsets (blocks) of V that satisfy

(1) If B ∈ B, then |B|∈K and

(2) Every pair of distinct elements of V occurs in exactly λ blocks of B. The integer λ is the index of the PBD. The notations PBD (v, K) and K-PBD of order v are often used when λ = 1.

Combining all these definitions one can summarily say that

A pairwise balanced design (or PBD) with index λ is a pair (V, B) where
1. V is a finite set of points,
2. B is a collection of subsets of V called blocks,
3. Every pair of distinct points of V occurs in exactly λ blocks.

For example: The following blocks constitute a PBD (10, {3, 4}, 1).
{1,2,3,4} {2,5,8} {3,5,0} {4,5,9}
{1,5,6,7} {2,6,9} {3,6,8} {4,6,0}
{1,8,9,0} {2,7,0} {3,7,9} {4,7,8}.
where, v = 10, b =3, k =4, λ =1, r = 3

Parameters of PBDs

v (order) is the size of V (elements of V are points, varieties or treatments)
b (block numbers) is the number of elements of B (elements of B blocks)
r (replication numbers) is the number of blocks to which every point belongs
k (blocks size) is the common size of each block and λ (index) is the number of blocks to which every pair of distinct point belongs.
According to Li ([Li99]) An LD(n,k,p,t;b) is a Lotto Design for a set of k-sets (blocks) of an n-set such that any p-set intersects at least one k-set in t or more elements. For example the following constitute blocks of an LD(13,6,5,3;5)
{1,2,3,4,5,6},{1,2,3,4,5,7}.{1,2,3,4,6,7},
{1,2,3,5,6,7},{8,9,10,11,12,13}
Here, n =13, k = 6, p = 5, t = 3 and b = 5

The Lottery Game

Li (1999) described a typical lottery the following way: For a small fee, a person chooses k numbers from n numbers or has the numbers chosen randomly for him. This constitutes the ticket. The sale of tickets is stopped at a certain point and the government or casino picks p numbers from the n numbers randomly. These p numbers are called the winning numbers. If any of the tickets sold match t or more of the winning numbers, a prize is given to the holder of the matching ticket. The larger the value of t, the larger the prize. Usually, t must be three or more to receive a prize.
Li (1999) gave the following definition for Lotto Designs.
Suppose $n, k, p\ and\ t$ are integers and B is a collection of K-subsets of a set X of v elements (usually X is$X\ (n)$).Then$B$ is an (n, k, p, t) Lotto Design (LD) if an arbitrary p-subset of $X\ (n)$ intersects some K-set of Bin at least t-elements. The K-sets in $B$are known as the blocks of the Lotto design$B$. The elements of X are known as the varieties of the design. $L(n, k, p, t)$ denotes the smallest number of blocks in any (n, k, p, t)-lotto design.

Pairwise Balanced Designs (PBDs) Lotto Designs (LDs) are combinatorial using latter to construct the formal we be a breakthrough of this research work.

2. METHODOLOGY

Several methods abound for the construction of PBDs. Many of the methods utilizes various computer techniques. Also, so many existence and enumeration problems have been resolved due to the use of various computer techniques. New and large classes of designs have been developed and analyzed through the use of computer. Thus, the field of computer as earlier written has been playing an important role in design theory.
This work will be introducing another method, the use of lotto designs in the construction of PBDs. Computer techniques would be utilized in achieving our objectives. This is because these techniques save time and ensure accuracy. The problem will be divided into stages as listed below:
Stage one: We shall identify PBDs that qualify as LDs. A FORTRAN program based on appropriate computer algorithm will be written to identify the PBDs out of those that will be qualified will be randomly selected for the purpose of our research.
Stage two: some conditions will be imposed on the LDs that will be constructed so as to ensure their compliance with PBDs.
Stage three: construction of some specific LDs will be done from the parents PBDs. This will be achieved through the writing of another computer program that utilized the binary search of generating subsets of a specific universal set in the Microsoft office access database context.
In stage one, we obtained r and λ from each of the two selected parents PBDs (6,{2,3},1) and (6,{2,3},2). They were combined with p and t in the Li's equation so as to be to identify which of the PBDs would qualify ad LDs since the two selected PBDs showed characteristics of LDs. The two PBDs which qualified with specific Lotto parameters were selected.
The lotto parameters were obtained by combining the corresponding v and k from each PBD with the corresponding p and t in cases that have YES as the answer. Thus, we have the particular lotto design with parameters LD(v,k,p,t) for each case where:
p is the positive integer and it constitutes the winning numbers selected by the organizers of the lottery game.
t is a positive integer and it is the numbers in k that match the winning numbers p.
r - Number of replicates of elements of the particular PBDs
λ-the number of blocks of pair of elements of a PBD appear together.

v (order) is the size of V (elements of V are points, varieties or treatments) and is equivalent to the n numbers available for that particular lottery format.
k (blocks size) is the common size of each block and is equivalent to the k numbers a players chooses out of n numbers available in the lottery game.
Thus, The PBD(6,{2, 3}, 1) that qualify as LD(n,k p,t) when **λ=1** are:
LD(6,3,4,3), LD(6,3,5,3), LD(6,3,5,4), LD(6,3,6,3), LD(6,3,6,4), LD(6,3,7,3), LD(6,3,7,4), LD(6,3,7,5), LD(6,3,8,3), LD(6,3,8,4), LD(6,3,8,5), LD(6,3,9,3), LD(6,3,9,4), LD(6,3,9,5), LD(6,3,9,6), LD(6,3,10,3), LD(6,3,10,4), LD(6,3,10,5), LD(6,3,10,6), LD,(6,3,11,3), LD(6,3,11,4), LD(6,3,11,5), LD(6,3,11,6), LD(6,3,11,7) and The PBD(6,{2,3},2) that qualify as LD(n,k,p,t) when λ =2 are: LD(6,3,3,3), LD(6,3,4,3), LD(6,3,4,4), LD(6,3,5,3), LD(6,3,5,4), LD(6,3,5,5), LD(6,3,6,3), LD(6,3,6,4), LD(6,3,6,5), LD(6,3,6,6), LD(6,3,7,3), LD(6,3,7,4), LD(6,3,7,5), LD(6,3,7,6), LD(6,3,7,7), LD(6,3,8,3), LD(6,3,8,4), LD(6,3,8,5), LD(6,3,8,6), LD(6,3,8,7), LD(6,3,8,8), LD(6,3,9,3), LD(6,3,9,4), LD(6,3,9,5), LD(6,3,9,6), LD(6,3,9,7), LD(6,3,9,8), LD(6,3,10,3), LD(6,3,10,4), LD(6,3,10,5), LD(6,3,10,6), LD(6,3,10,7), LD(6,3,10,8), LD(6,3,11,3), LD(6,3,11,4), LD(6,3,11,5), LD(6,3,11,6), LD(6,3,11,7), LD(6,3,11,8).
By Imposing conditions $k = t$ and $p \leq v$ on all the LDs. This imposition is based on the observation of the LDs taking into consideration the theory of lotto designs and the theory of combinatorial analysis. If $p \geq v$, the combination $\binom{v}{p}$ will not be defined. Also, from observations of the LD s, all the *ts* were equal to the *ks* in their parent PBDs.
Thus, The PBD(6,{2,3},1), produced the following LDs: (6,3,4,3), (6,3,5,3), (6,3,6,3), (6,3,7,3), (6,3,8,3), (6,3,9,3), (6,3,10,3), (6,3,11,3) and The PBD(6,{2,3},2), produced the following LDs: (6,3,3,3), (6,3,4,3), (6,3,6,3), (6,3,7,3), (6,3,8,3), (6,3,9,3), (6,3,10,3), (6,3,11,3).
For each of the parents PBDs used to produced LDs we then test the LDs produced so as to ensure conformity to the underlying properties of PBDs, this is done by using the second software**.**

The constructions when λ =1

For PBD(6,{2,3},1) and LD(6,3,4,3)
Report on Intersections Comparison Value t = 3
p = 1 2 3 4
1 2 3
1 2 4
1 3 4
2 3 4
Hence, this produced a new PBD(4,{3,4},1)

For PBD(6,{2,3},1) and LD (6,3,5,3)
Report on Intersections Comparison Value t = 3
p = 1 2 3 4 5
1 2 3 1 3 5 2 4 5
1 2 4 1 4 5 3 4 5
1 2 5 2 3 4
1 3 4 2 3 5
Hence, this produced a new PBD(5,{3,10},1)

For PBD(6,{2,3},1) and LD (6,3,6,3)
Report on Intersections Comparison Value t = 3
p = 1 2 3 4 5 6
1 2 3 1 3 5 2 3 5 3 4 5
1 2 4 1 3 6 2 3 6 3 4 6
1 2 5 1 4 5 2 4 5 3 5 6
1 2 6 1 4 6 2 4 6 4 5 6
1 3 4 1 5 6 2 5 6
Hence, this produced a new PBD(6,{3,19},1).

The constructions when λ = 2

For PBD(6,{2,3},2) and LD(6,3,3,3),
Report on Intersections Comparison Value t = 3
p = 1 2 3
1 2 3
Therefore this produced a new PBD(3,{3,1},1)

For PBD(6,{2,3},2) and LD(6,3,4,3)
Report on Intersections Comparison Value t = 3
p = 1 2 3 4
1 2 3
1 2 4
1 3 4
2 3 4
Hence, this produced a new PBD(4,{3,4},1).

CONCLUSION

Pairwise Balanced Designs (PBDs) are of fundamental importance in combinatorial theory they are of interest in their own right, and have many applications in the constructions of other types of designs while Lotto Designs (LDs) are also combinatorial. From our research work it has been established that PBD(6,{2,3},1) produced LD(6,3,4,3), (6,3,5,3) and (6,3,6,3) when λ =1 and when, λ =2. PBD(6,{2,3},1) produced LDs(6,3,3,3), (6,3,4,3), and (6,3,6,3).
For each of the parents PBDs used to produced LDs we then used each of the LDs to construct a new PBDs(4,{3,4},1), (5,{3,10},1) and (6,{3,19},1) and PBDs(3,{3,1},1), (4,{3,4},1) and (6,{3,19},1) respectively.

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