MODELLING EXTINCTION OF POLIO TRANSMISSION AGENTS BY STOCHASTIC DIFFERENTIAL EQUATIONS

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ABSTRACT: Eradicating poliomyelitis has been a major health concern to stakeholders the world over. In this study, we have developed a pair of stochastic equations for the extinction probability functions in two types of human transmission agents namely: infectives (types 1) and carriers (type 2). By solving the equations, the result shows that with only one asymptomatic carrier in a population, eradication is possible when $(\beta_{22} + \beta_{21}\_\mu)^2 > 4\beta_{22}\mu$ where β_{ij} is the Poisson rate at which type i transmission agent produces type j transmission agent and μ is the mean infectious period of the asymptomatic carriers.

KEYWORDS: Extinction Probability, Polio Free Certification, Transmission Agents, Poisson Rate, Conditional Probability.

INTRODUCTION

In the past two decades, the trend of poliomyelitis transmission in some countries of the world has been characterized by decline in incidence rate, near eradication and sudden re-appearance of the disease. For example, Nigeria had about 35,000 reported polio cases in 1988 but due to effective control strategies, the number of cases dropped to 359 in 2014. By 2015, Nigeria was removed from the list of polio endemic countries by World Health Organization (WHO). It therefore needed a no-incidence period of 36 months to be certified polio free by WHO. But by August 2016, when the country was just 23 months away from being given a clean bill of health, the disease re-emerged with 2 recorded cases in the North- East of the country. The concern now is about how Nigeria will get rid of polio virus completely before September 2019 which is the new set time for its polio free certification.

In India, the presence of asymptomatic polio virus in older children and adults is suggestive of a cause of the recurrence of the disease in the northern part of the country and the need to expand the scope of anti-polio vaccination to include adults. ([MVJ10]). There have been many mathematical researches on poliomyelitis. Raiimandad et al ([RHT11]) used a differential equation based model to develop an individual based model for polio transmission. They concluded that the choice of network structure determines the model estimate of cases and the dynamics of the outbreak. Also, Argawal and Bhadauria in 2016 ([AB16]) studied the spread of polio in a population with variable size structure.

Olawuwo and Ugbebor ([OU16]) wrote a model on re-emergence of polio in Nigeria with probability 0.0322 while J. E. Thoumpson ([Tho14]) developed a bilinear mathematical model of ordinary differential equation to investigate the transmission dynamics of polio virus in Nigeria.

The human enterovirus called polio virus is transmitted through the oral-fecal route by 2 types of agents namely: infective (type 1) and asymptomatic carriers (type 2). While infectives show clear signs of polio as they transmit it, the asympotomatic carriers do not. They transmit the disease unnoticed.

The disease can be eradicated when the two types of transmission agents cease to exist. The aim of this work was to find the probability function for the extinction of the two transmission agents.

At any time t, the number of individuals diagnosed with polio virus is the process X(t). An infective is assumed to produce an infective at the rate β_{11} and a carrier at the rate β_{12}. Also, a carrier is assumed to produce an infective at the rate β_{21} and a carrier at the rate β_{22}.

METHODOLOGY

We represent the number of symptomatic infectives by $I_s(t)$ and the number asymptomatic carriers at time t by $I_c(t)$ where

$$I_s(0) = s \text{ and } I_c(0) = c \tag{1}$$

Let the extinction probability of the 2 transmission agents be defined by the conditional probability function.

$$q_{s,c}(t) = P\{X(t) = X(0)\ and\ I_s(t) + I_c(t) = 0|\big(I_s(0), I_c(0)\big) = (s,c)\} \tag{2}$$

where X(t) is the number of individuals diagnosed with polio symptoms at time t

The infectious contacts are assumed to be independent at Poisson process rate $\beta_{s,c}$. Asymptomatic carriers have mean infectious period μ and and infectives have mean infectious period γ. Hence, we derive backward equations for the function $q_{s,c}(t)$ as follows:

We find the conditional probability of no new discovery and no transmission agent given that (s, c) = (0, 1) or (1, 0) at time t + Δt.

For (s,c) = (1,0), we have

$$q_{1,0}(t+\Delta t) = \beta_{1,1}\Delta t q_{2,0}(t) + \beta_{1,2}\Delta t q_{1,1}(t) + [1-\Delta t(\beta_{1,1}+\beta_{1,2}+\gamma)]q_{1,0}(t) + o(\Delta t) \quad (3)$$

But

$$q_{2,0}(t) = q_{1,0}(t)q_{1,0}(t) \text{ and } q_{1,1}(t) = q_{1,0}(t)q_{0,1}(t) \quad (4)$$

Therefore,

$$q_{1,0}(t+\Delta t) = \beta_{1,1}\Delta t q_{1,0}^2(t) + \beta_{1,2}\Delta t q_{1,0}(t)q_{0,1}(t) + [1-\Delta t(\beta_{1,1}+\beta_{1,2}+\gamma)]q_{1,0}(t) \quad (5)$$

$$\frac{q_{1,0}(t+\Delta t)-q_{1,0}(t)}{\Delta t} = \beta_{1,1}q_{1,0}^2(t) + \beta_{1,2}q_{1,0}(t)q_{0,1}(t) - (\beta_{1,1}+\beta_{1,2}+\gamma)q_{1,0}(t) + o(\Delta t) \quad (6)$$

Taking limit as Δt → 0,

$$\lim_{\Delta t\to 0}\frac{q_{1,0}(t+\Delta t)-q_{1,0}(t)}{\Delta t} = \beta_{1,1}q_{1,0}^2(t) + \beta_{1,2}q_{1,0}(t)q_{0,1}(t) - (\beta_{1,1}+\beta_{1,2}+\gamma)q_{1,0}(t)$$

$$\frac{dq_{1,0}(t)}{dt} = \beta_{1,1}q_{1,0}^2(t) + \beta_{1,2}q_{1,0}(t)q_{0,1}(t) - (\beta_{1,1}+\beta_{1,2}+\gamma)q_{1,0}(t) \quad (7)$$

For (s, c) = (0, 1), we have

$$q_{0,1}(t+\Delta t) = \beta_{22}\Delta t q_{0,2}(t) + \beta_{2,1}\Delta t q_{1,1}(t) + [1-\Delta t(\beta_{2,2}+\beta_{2,1}+\mu)]q_{0,1}(t) + \mu\Delta t \quad (8)$$

$$q_{0,1}(t+\Delta t) - q_{0,1}(t) = \beta_{2,2}\Delta t q_{0,2}(t) + \beta_{2,1}\Delta t q_{1,1} - \Delta t(\beta_{2,2}+\beta_{2,1}+\mu)q_{0,1}(t) + \mu\Delta t \quad (9)$$

$$= \beta_{2,2}\Delta t q_{0,1}^2(t) + \beta_{2,1}\Delta t q_{1,0}(t)q_{0,1}(t) - \Delta t(\beta_{2,2}+\beta_{2,1}+\mu)q_{0,1}(t) + \mu\Delta t \quad (10)$$

$$\frac{q_{0,1}(t+\Delta t)-q_{0,1}(t)}{\Delta t} = \beta_{2,2}q_{0,1}^2(t) + \beta_{2,1}q_{1,0}(t)q_{0,1}(t) - (\beta_{2,2}+\beta_{2,1}+\mu)q_{0,1}(t) \quad (11)$$

As Δt → 0

$$\lim_{\Delta t\to 0}\frac{q_{0,1}(t+\Delta t)-q_{0,1}(t)}{\Delta t} = \beta_{2,2}q_{0,1}^2(t) + \beta_{2,1}q_{0,1}(t) - (\beta_{2,2}+\beta_{2,1}+\mu)q_{0,1}(t) + \mu \quad (12)$$

Finally,

$$\frac{dq_{0,1}(t)}{dt} = \beta_{2,2}q_{0,1}^2(t) - \beta_{2,1}q_{1,0}(t)q_{0,1}(t) - (\beta_{2,2}+\beta_{2,1}+\mu)q_{0,1}(t) + \mu \quad (13)$$

The 2 equations (7) and (13) are respectively subject to the initial condition $q_{1,0}(0) = 0$ and $q_{0,1}(0) = 0$. This is because extinction will be impossible without identifying the initial transmission agent called index case. Thus, the two conditions jointly imply that $q_{1,0}(t) = 0$, which apart from being the solution of equation (7) also reduces equation (13) to the initial value problem (14)

$$q_{0,1}'(t) = \beta_{2,2}q_{0,1}^2(t) - (\beta_{2,2}+\beta_{2,1}+\mu)q_{0,1}(t) + \mu,\ q_{0,1}(0) = 0 \quad (14)$$

Now, solving equation (14) for $q_{0,1}(t)$;

$$\frac{dq_{0,1}(t)}{\beta_{2,2}q_{0,1}^2(t) - (\beta_{2,2}+\beta_{2,1}+\mu)q_{0,1}(t) + \mu} = dt \quad (15)$$

Let α_1 and α_2 be the zeros of the polynomial

$$\phi(t) = \beta_{2,2}q_{0,1}^2(t) - (\beta_{2,2}+\beta_{2,1}+\mu)q_{0,1}(t) + \mu$$

$$\Rightarrow \alpha_1 = \frac{(\beta_{2,2}+\beta_{2,1}+\mu) + \sqrt{(\beta_{2,2}+\beta_{2.1}+\mu)^2 - 4\beta_{2,2}\mu}}{2\beta_{2,2}} \quad (16)$$

$$\alpha_2 = \frac{(\beta_{2,2}+\beta_{2,1}+\mu) - \sqrt{(\beta_{22}+\beta_{2,1}+\mu)^2 - 4\beta_{2,2}\mu}}{2\beta_{2,2}} \quad (17)$$

From (15);

$$\frac{dq_{0,1}(t)}{\beta_{2,2}(q_{0,1}(t)-\alpha_1)(q_{0,1}(t)-\alpha_2)} = dt \quad (18)$$

Integrating (18)

$$\int \frac{dq_{0,1}(t)}{(q_{0,1}(t)-\alpha_1)(q_{0,1}(t)-\alpha_2)} = \int \beta_{2,2}dt \quad (19)$$

$$\Rightarrow \int\left\{\frac{1}{(\alpha_1-\alpha_2)(q_{0,1}(t)-\alpha_1)} + \frac{1}{(\alpha_2-\alpha_1)(q_{0,1}(t)-\alpha_1)}\right\}dq_{0,1}(t) = \beta_{2,2}t + C \quad (20)$$

$$\frac{1}{\alpha_1 - \alpha_2}\ln\left(q_{0,1}(t) - \alpha_1\right) + \frac{1}{(\alpha_2 - \alpha_1)}\ln\left(q_{0,1}(t) - \alpha_2\right) = \beta_{2,2}t + C$$

$$(\alpha_2 - \alpha_1)\ln\left(q_{0,1}(t) - \alpha_1\right) + (\alpha_1 - \alpha_2)\ln\left(q_{0,1}(t) - \alpha_2\right) = (\alpha_1 - \alpha_2)(\alpha_2 - \alpha_1)(\beta_{2,2}t + C)$$

$$\ln\left\{\frac{q_{0,1}(t) - \alpha_1}{q_{0,1}(t) - \alpha_2}\right\} = (\alpha_1 - \alpha_2)(\beta_{2,2}t + C) \qquad (21)$$

But $q_{0,1}(0) = 0$, hence

$$\ln\left[\frac{\alpha_{1\alpha}}{\alpha_2}\right] = (\alpha_1 - \alpha_2)C$$

So,

$$C = \frac{1}{\alpha_1 - \alpha_2}\ln\left(\frac{\alpha_1}{\alpha_2}\right) \qquad (22)$$

Thus,

$$\ln\left\{\frac{q_{0,1}(t) - \alpha_1}{q_{0,1}(t) - \alpha_2}\right\} = (\alpha_1 - \alpha_2)\left[\beta_{2,2}t + \frac{1}{\alpha_1 - \alpha_2}\ln\frac{\alpha_1}{\alpha_2}\right] \qquad (23)$$

$$= (\alpha_1 - \alpha_2)\beta_{2,2}t + \ln\frac{\alpha_1}{\alpha_2}$$

$$\ln\left\{\frac{q_{0,1}(t) - \alpha_1}{q_{0,1}(t) - \alpha_2}\right\} - \ln\frac{\alpha_1}{\alpha_2} = (\alpha_1 - \alpha_2)\beta_{2,2}t \qquad (24)$$

$$\ln\frac{\alpha_2}{\alpha_1}\left[\frac{q_{0,1}(t) - \alpha_1}{q_{0,1}(t) - \alpha_2}\right] = (\alpha_1 - \alpha_2)\beta_{2,2}t$$

$$\frac{\alpha_2}{\alpha_1}\left[\frac{q_{0,1}(t) - \alpha_1}{q_{0,1}(t) - \alpha_2}\right] = e^{(\alpha_1 - \alpha_2)\beta_{2,2}t} \qquad (25)$$

$$\frac{q_{0,1}(t) - \alpha_1}{q_{0,1}(t) - \alpha_2} = \frac{\alpha_1}{\alpha_2}e^{(\alpha_1 - \alpha_2)\beta_{2,2}t} \qquad (26)$$

$$\alpha_2(q_{0,1}(t) - \alpha_1) = \alpha_1(q_{0,1}(t) - \alpha_2)e^{(\alpha_1 - \alpha_2)\beta_{2,2}t} \qquad (27)$$

$$\left(\alpha_2 - \alpha_1 e^{(\alpha_1 - \alpha_2)\beta_{2,2}t}\right)q_{0,1}(t) = \alpha_1\alpha_2 - \alpha_1\alpha_2 e^{(\alpha_1 - \alpha_2)\beta_{2,2}t} \qquad (28)$$

$$q_{0,1}(t) = \frac{\alpha_1\alpha_2 - \alpha_1\alpha_2 e^{(\alpha_1 - \alpha_2)\beta_{2,2}t}}{\alpha_2 - \alpha_1 e^{(\alpha_1 - \alpha_2)\beta_{2,2}t}} \qquad (29)$$

$$q_{0,1}(t) = \frac{\alpha_1\alpha_2\left(1 - e^{(\alpha_1 - \alpha_2)\beta_{2,2}t}\right)}{\alpha_2 - \alpha_1 e^{(\alpha_1 - \alpha_2)\beta_{2,2}t}} \qquad (30)$$

RESULTS AND CONCLUSION

We have derived a pair of stochastic differential equation for the extinction of 2 types of polio transmission agents namely: Symptomatic infective and asymptomatic carriers in a population at time t. The solution of the equation depends not only on time but also on the parameters α_1 and α_2, ($\alpha_1 > \alpha_2$) which are the zeros of the polynomial $\phi(t)$. These zeros are real only when $(\beta_{22} + \beta_{21} + \mu)^2 > 4\beta_{22}\mu$.
In other words, with only one asymptomatic carrier and no symptomatic polio transmission agent in a population, eradication is possible when $(\beta_{2,2} + \beta_{2,1} + \mu)^2 > 4\beta_{2,2}\mu$.

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