CONSTRUCTION OF CONGRUENT CLASSES OF PAIRWISE BALANCED DESIGNS USING LOTTO DESIGNS

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ABSTRACTS: Among the Incomplete Block Designs (IBDs), Balanced Incomplete Block Designs (BIBDs) are mostly studied. However, BIBD is not available for all parameters of most designs therefore limited in applications. So, in place of BIBD there is need for another incomplete block design that could be used for varieties of applications. Thus, the need for Pairwise Balanced Designs (PBDs). PBD (n, K, λ) is a block design where n is the number of treatments, where K= {k_1, k_2…, k_b} is the set of sizes of block and λ is number of time a pair of treatment appear together within blocks. Also, little is known about the construction of PBDs using Lotto Designs (LDs). An LD(n, k, p t) is a set of k-blocks of an n-treatments such that any p-sets intersect at least one k-block in t number of times. The aim of the study is to provide a simple method for constructing two classes of PBD(n, K, λ) when = K {3, 4} or {3, 4, 5} using appropriate LDs; establish conditions for the identification of LDs that could use to construct the classes of PBDs; and derive theorems and simple steps for the construction of PBDs from LDs.

The research work utilized the Li's inequality $\left\lfloor \frac{pr}{t-1} \right\rfloor \binom{t-1}{2} + \binom{pr-\left\lfloor \frac{pr}{t-1} \right\rfloor (t-1)}{2} < \binom{p}{2}\lambda$ to obtain LDs that are PBDs using r and λ obtained from the classes of PBDs on the Li inequality. Some LDs were generated and based on the structure of the classes of the PBDs to be constructed, some conditions were imposed. Hence some LDs were found to qualify as PBDs. Theorems were proposed and proved. Hence, the following results were obtained: Any LD(n, k, p, t) satisfying the conditions: $3 \leq k \leq 5$ and $n = p$ qualify as PBDs and the proposed theorems were: (a) 2-LDs(n, 3, p, 3), (n, 4, p, 4) is a PBD(n, {3, 4}) if and only if n ≡ 0, 1 (mod 3); (b). 3-LDs(n, 3, p, 3), (n, 4, p, 4), (n, 5, p, 5) is a PBD(n, {3, 4, 5}) if and only if n ≡ 2, 3 (mod 4). Thus, two classes of PBDs(n, K, λ) when K= {3, 4}) and K= {3, 4, 5} were constructed from the appropriate LDs for all admissible n-treatments satisfying the specified conditions.

KEYWORDS: *Block Designs, Incomplete Block Designs (IBD), Balanced Incomplete Block Designs (BIBDs), Pairwise Balanced Designs (PBDs) and Lotto Designs (LDs).*

1. INTRODUCTION

Design theory has its roots in recreational mathematics. Many types of designs that are studied today were first considered in the context of mathematical puzzles or brain-teasers in the eighteenth and nineteenth centuries. The study of design theory as a mathematical discipline really began in the twentieth century due to applications in the design and analysis of statistical experiments. Designs have many other applications as well, such as tournament scheduling, lotteries, mathematical biology, algorithm design and analysis, networking, group testing, and cryptography ([Sti03]). Combinatorial design theory is the study of arranging elements of a finite set into patterns (subsets, arrays etc.) according to specified rules. Its history dates back to Euler's work on Latin squares, but it mostly gained recognition after Ronald Fisher, a statistician and geneticist, developed techniques for the design and analysis of experiments. Today it has become a fast-growing subfield of mathematics with close ties to several other areas including graph theory, coding theory, cryptography and engineering applications.

1.1. The Block Designs

In block designs, blocks are built from the set of X (treatments) in which each element of X is called treatment, blocks are formed when pairs of treatments are chosen as subsets to satisfy some set of properties that are deemed useful for a particular application. The history of block Designs is as old as the history of Steiner systems dates back to 1733, when W.S.B. Woolhouse asked the following question: how do we arrange Fifteen young ladies who walk out three abreast for seven days in succession if no two shall walk abreast? this question was later known as Kirkman's schoolgirl problem posted in Lady's and Gentlemen's Diary of 1844. The problem was solved in 1847 by Rev. T.P. Kirkman ([Kir47]). Woolhouse studied triple systems, i.e., block designs with k = 3. Later in 1853, Steiner discussed t -designs with k = t + 1 and λ = 1. The actual study of block designs started with a study of algebraic curves by Plucker ([Plu35]). He encountered a Steiner Triple System (STS) of order 9 and claimed that an STS could exist only when m≡ 3 (mod 6). He correctly revised this condition to m ≡ 1, 3(mod 6) in 1839. In block design, a design is a pair (X, Ɓ), where X is a set of some elements called treatments, and Ɓ is a collection of some subsets of X called blocks. The numbers of treatments and blocks are denoted by n = n(X) and b = n(Ɓ) respectively. A block design is an incidence system (n, k, λ, b, r) where set X of n- treatments are partitioned into a family Ɓ of b subsets (blocks) in such a way that any two treatments determine λ blocks, with k size in each block, and each treatment is contained in r different blocks. When k < n, the design is said to be incomplete e.g BIBD. When k = n- a case of Randomized Completely Block Designs (RCBD), and where K={ k_1, k_2,…k_b} a case of Pairwise Balanced Designs.

1.2. The Pairwise Balanced Design

Wilson ([Wil71a, Wil71b]) defined a Pairwise Balanced Design (PBD) of order n as a pair (X, Ɓ) where Ɓ is a set of cardinality B, and B is a set of subsets of X (each of which is called a block) with the property that every 2-treatments subset of X is contained in a unique block.

For example: a PBD(5, {2, 3}, 1), where X= {1, 2, 3, 4, 5}, K={2, 3}, λ= 1, has the following set of blocks:

Ɓ =.{{1, 2, 3}, {1, 4, 5}, {2, 4}, {2, 5}, {3 ,4}, {3, 5}}

Where:

n is the total number of the treatment in the design therefore n = 5 and n = n(X),

K is the set of sizes of the block of the design.

Therefore, K = {2, 3}, clearly K = {k_1..., kb},

Where:

k_i is the block size as i= 1, 2, …, b, therefore k_1 = 2 and k_2 = 3

b_i is the number of units in the blocks size of k_i,

Therefore, b_1= 2 and b_2 = 4

Clearly b = $\sum_i bi$

λ is the number of time a pair of treatment appears together within blocks, therefore λ = 1

In a Pairwise Balanced Design when λ = 1, it is usually written as PBD(n, K).

Ɓ is the collection of all the set of blocks (vector space) while B is the each of the block inside the set cardinality (vector).

Therefore Ɓ ={{B_1, B_2…B_b}} and b is the number of blocks for each of the block sizes.

Among all the block designs, Balanced Incomplete Block Design (BIBD) is the easiest in terms of construction and analysis. However, BIBD is not available for all parameters of most designs, BIBD exist only if:

a). nr = bk;

b). λ (n – 1) = r (k – 1);

and

c). b ≥ n

So, in place of BIBD, there is need for another block design which is balanced, variance balanced and flexible, design that could be used for varieties of applications such as construction of other important designs, confirmation of the existence of other designs etc. thus the need for Pairwise Balanced Designs (PBDs). Bose ([Bos39]) introduced the present terminology of balanced incomplete block designs although the present use of n, b, r, k, λ was due to Yates except that he originally used t for treatment. In other words, one can say that Bose and Shrikhande ([BS59a, BS59b, BS60a, BS60b]) introduced a general class of incomplete block design which they called pair-wise balanced design of index λ. Pairwise Balanced Design also shared some of the properties of BIB designs like every pair of treatments occurs together in λ blocks ([Wil71a, Wil71b]). The concept of Pairwise Balanced Design is merely the combinatorial interest in block designs. Because with the help of pair-wise balanced design many other incomplete block designs can be constructed ([Wil72a, Wil72b, Wil72c, Wil72d]). Bose and Shrikhande ([BS59a, BS59b]) discussed about the pairwise balanced design in addition to existence of orthogonal Latin square designs. Bose and Shrikhande ([BS60a, BS60b]) obtained the various methods for the construction of pair-wise orthogonal sets of Latin square design. However, the detailed discussion and construction on pair-wise balanced design is studied by Bose and Shrikhande ([BS60a, BS60b]). In other words, we can say that Bose and Shrikhande ([BS59a, BS59b, BS60a, BS60b]) introduced a general class of incomplete block design which they called pair-wise balanced design of index λ. The pair-wise balanced design also shared some of the properties of BIB designs like every pair of treatments occurs together in λ blocks. The concept of Pairwise Balanced Design is merely the combinatorial interest in block designs. Because with the help of Pairwise Balanced Design many other incomplete block designs can be constructed. Smith, Blaikie and Taylor ([SBT98]) proved the existence of a PBD(n, K, λ) with bi blocks of size $k_i \in K$ and went further to present useful expression that connects PBD parameters with number of blocks of different sizes as follows:

$$\lambda n(n-1) = \sum_i biki\ (ki-1)$$

Clearly, b $=\sum_i bi$.
The application of PBDs in the construction of related combinatorial systems is of paramount important in the design theory. PBDs were used in the construction of most of resolvable and recursive designs such as: Varieties of Short Conjugate, Orthogonal Quasi-Group Identities, Orthogonal arrays with interesting Conjugacy Properties, Egde-coloured Designs, and Mendelsohn Designs etc. Bose and Shrikhande ([BS60a, BS60b]) used pairwise balanced design to construct Mutually Orthogonal Latin Square (MOLS). Hedayat and Stufken ([HS89]) showed that the problems of constructing pairwise balanced designs and variance balanced block designs are equivalent. Effanga, Ugboh, Enang and Eno ([E+09]) developed a non-linear non-preemptive binary integer goal programming model for the construction of *D*-optimal pairwise balanced incomplete block designs. PBDs can also be used to construct other designs such as Steiner Triple Systems (STS), Pairwise Additive Designs PBIB etc. It also has greater significance in the application to the solution of existence questions for other types of designs such as t-Designs, Partially Balanced Designs etc. ([IB03]).

1.3. Existence theory of Pairwise Balanced Designs

One of the most important breakthroughs in design theory was made by Richard Wilson in the early 1970s. He showed that the trivial necessary conditions for the existence of various kinds of designs are asymptotically sufficient theory by Wilson. Wilson ([Wil71a, Wil71b]) proved the existence of a PBD(n, K), where n is the number of treatments and K is the set of blocks where $k \in K$ and $k \geq 2$ ($K \subseteq Z_{\geq 2}$) which means n treatments contains the allowed block sizes. Since the set of blocks incident with any treatment must contain each other treatment once, and since the set of pairs of treatment must partition into the pairs covered in each block, thus, the 'divisibility' conditions:

$$\alpha(K) \mid n-1 \qquad (1)$$

and

$$\beta(K) \mid n(n-1) \qquad (2)$$

where $\alpha(K) := \gcd\{k-1 : k \in K\}$ and
$\beta(K) := \gcd\{k(k-1) : k \in K\}$.
(note: gcd means greater common divisor and $Z_{\geq 2}$ means integer greater than or equal to 2*).*
The integers n satisfying (1) and (2) are *admissible.* Wilson (1972) states that admissibility is sufficient for existence of a PBD(n, K), provided n is large and went on to prove it(see the main work).

For example,
Suppose $K = \{3, 4, 6\}$.
Then it is easy to compute

$$\alpha(K) = \gcd\{2, 3, 5\} = 1 \qquad (3)$$

and

$$\beta(K) = \gcd\{6, 12, 30\} = 6$$

According to (1) and (2), necessary conditions for the existence of a PBD with block sizes 3, 4 or 6 are that:

n-1 ≡ 0(mod 1) and (n-1) ≡ 0(mod 6)

The first conditions just say n is an integer. The second condition is satisfied iff n(n-1) ≡ 0 (mod 3), so we have n≡ 0 or 1 mod 3. Then n ≥ 3 follows since 3 is the smallest block size.

1.4. The Lotto Designs

According to Stinson ([Sti03]) an (n, k, p t) lotto designs is a set of k-set (blocks) of an n -set (treatments) such that any p-set interest at least one k-set (blocks) in t or more times. Suppose n, k, p and t are integers and B is a collection of k-subsets of a set X of n treatments (usually X is X(n)). Then B is an (n, k, p, t) Lotto Designs (LD) if an arbitrary p-subsets of X(n) intersects relevant k-set of B in at least t times. The k-sets in B are known as the blocks of the Lotto Designs. The elements X are known as the n -treatments of the design. Lotto design can also be denoted by (X, B) Where B denotes the blocks of the design and X denotes the set from which the treatments of the blocks of B are chosen.
For example,
An LD(13, 6, 5, 3) is a typical example of a Lottery Design, where:
n =13, k = 6, p = 5 and t = 3 has the following set of blocks:
{ {1, 2, 3, 4, 5, 6},
{1, 2, 3, 4, 5, 7},
{1, 2, 3, 4, 6, 7},
{1, 2, 3, 5, 6, 7},
{8, 9, 10, 11, 12, 13} }

2. METHODOLOGY

Numerous pairwise balanced designs constructed so far by various scholars were based on the understanding of Group Divisible Designs (GDDs) which have numerous analogues, interpretations and meanings. However, PBD plays important roles in the designs theory. More importantly, among the incomplete block designs PBDs have varieties of applications. Therefore, this research work introduces a new method for the construction of Pairwise Balanced Designs (PBDs) using Lotto

Designs (LDs). In designs theory, if a new design is constructed from another related design it is known as a recursive construction and if a specific fitted class of designs are generated with specific conditions imposed using mathematical relations of modulus system the design is known as congruent class, therefore, this research work presents a simple way of construction of PBDs from an appropriate LDs which is both recursive and congruent. The next stages shall be followed:

Stage One: Specification of the classes of the PBDs to be constructed using developing theorems.

Stage Two: Identification of LDs that will qualify as PBDs. Computer program based on the Li inequality:

$$\left\lfloor \frac{pr}{t-1} \right\rfloor \binom{t-1}{2} + \binom{pr-\left\lfloor \frac{pr}{t-1} \right\rfloor}{2} (t-1) < \binom{p}{2}\lambda$$

was written so as to identified LDs that could produce two classes of the PBDs to be constructed.

Stage Three: Specify certain conditions and imposed those conditions on all the generated LDs from the Li inequality and Select all LDs that satisfy the specified conditions.

Stage Four: Derive theorems for the construction of the classes of PBDs(n, {3, 4}) and (n, {3, 4, 5}) from LDs.

Stage four: The actual construction.

2.1 Dimension of the set of blocks of PBDs

The dimension of a linear space is the maximum integer d such that any set of d points generates a proper subspace. For instance, the subspace generated by any two points is the line containing them. So, every nontrivial linear space has dimension at least two. Linear spaces have another name in the context of designs.

For Example, A (PBD) of index unity is a pair (X, Ɓ) where X is a set (of *points)* and Ɓ a class of subsets B of X (called *blocks)* such that any pair of distinct points of X is contained in exactly one of the blocks of Ɓ (and we may also require /B/ ≥ 2 for each B ϵ Ɓ). Such systems are also known as linear spaces. The *dimension* of a PBD is the maximum integer *d* such that any set of *d* points generates a proper flat. This definition is taken from the context of linear spaces. So every PBD(*n*, *K*) that has more than one block has dimension at least two. Therefore, $K \subseteq Z_{\geq 2}$ which means n treatments may contains the allowed block sizes. Specifically, this research work attempts to construct two classes of PBDs where, *K* = {3, 4}; and K= {3,4,5}.

3. THE CONSTRUCTION

The aim of using the Li-Inequality is to identify LDs that could qualify as PBD(n, K) where K = {3, 4} and {3, 4, 5} because it will be easier to use such LDs to construct PBDs. Li (1999) suggested that for any designs to be qualified as an LDs, such a design must have satisfied the bellow inequality:

$$\left\lfloor \frac{pr}{t-1} \right\rfloor \binom{t-1}{2} + \binom{pr-\left\lfloor \frac{pr}{t-1} \right\rfloor}{2} (t-1) < \binom{p}{2}\lambda$$

For A PBD with a set of K = {3, 4}

3.1. Preliminary Theorem 1

There exists a PBD(n, {3, 4}) of dimension two if and only if $n \equiv 0, 1 \pmod 3$ *provided* $n \geq 6$

Here, r = 3 and λ =1.

Hence, the followings LDs were generated:

n	k	p	t
5	3	5	3
5	4	5	3
6	3	6	3
6	3	6	4
7	3	7	3
7	4	7	4
8	4	8	3
9	3	9	3
9	4	9	4
10	3	10	3
10	4	10	4
10	5	10	5
11	3	11	3
11	4	11	4

3.2. Conditions imposed

From the above result, some LDs were produced from using λ = 1 and r = 3 obtained from the classes of PBDs(n, {3, 4}) on the Li inequality, this shows that some LDs were qualified as PBDs.

Thus, the conditions:

i). $3 \leq k \leq 4$ and

ii). n = p

which give the following LDs:

The LDs(6, 3, 6, 3), (6, 4, 6, 4)
The LDs(7, 3, 7, 3), (7, 4, 7, 4),
The LDs(9, 3, 9, 3), (9, 4, 9, 4)
The LDs(10, 3, 10, 3), (10, 4, 10, 4)
The LDs(12, 3, 12, 3), (12, 4, 12, 4)
The LDs(13, 3, 13, 3), (13, 4, 13, 4)
The LDs(15, 3, 15, 3),(15, 4, 15, 4)
The LDs(16, 3, 16, 3), (16, 4, 16, 4),
The LDs(18, 3, 18, 3), (18, 4, 18, 4)
The LDs(19, 3, 19, 3), (19, 4, 19, 4)
The LDs(21, 3, 21, 3), (21, 4, 21, 4)

The LDs(22, 3, 22, 3), (22, 4, 22, 4)
The LDs(24, 3, 24, 3),(24, 4, 24, 4)
The LDs(25, 3, 25, 3), (25, 4, 25, 4),
The LDs(27, 3, 27, 3), (27, 4, 27, 4)
The LDs(28, 3, 28, 3), (28, 4, 28, 4)
The LDs(30, 3, 30, 3), (30, 4, 30, 4)
The LDs(31, 3, 31, 3), (31, 4, 31, 4).
Etc.

All n-treatments satisfies: $n \equiv 0$ or $1 (\mathrm{mod} 3)$, when $n \geq 6$ and $\lambda = 1$. And the imposed conditions for any LDs to be qualified as PBDs:

i). $3 \leq k \leq 4$;
ii). n = p and
iii). $\lambda = 1$

Clearly, 2-LD(*n*, 3, p, 3), (*n*, 4 p, 4) = PBD(*n*, {3, 4}, 1).

3.3. Theorem 1

A PBD(n, {3, 4}) is a 2-LDs(n, 3, p, 3), (n, 4, p, 4) is if and only if the following conditions are satisfied:

i). for all $n \equiv 0, 1 \pmod 3$;
ii). $3 \leq k \leq 4$;
iiii). n = p and
iv). $\lambda = 1$

3.4. Derived Steps for construction of congruent class of PBD(*n*, {3, 4}, 1) from any 2- LDs(*n*, 3, p, 3), (*n*, 4, p, 4)

(1). Select any n-treatments that satisfy the condition $n \equiv 0, 1 \pmod 3$.
(2). Select any 2-LDs corresponding to n-treatments of the desired PBD that satisfy the followings conditions:

i). $3 \leq k \leq 4$;
ii). n = p and

(3). Then, generate the blocks of the PBDs by Partitioning n- treatments into sets of blocks of 3 or 4 such that each pair of treatments is contained in precisely one block which follows that $n(n - 1)$ is an integer linear combination of $k(k - 1)$, $k \in K$.

For Example

To construct a PBD(10, {3, 4}, 1), where, n-treatments is 10.
Therefore, X = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}.
Then, select 2- LDs(10, 3, 10, 3), (10, 4, 10, 4) that corresponds to the n-treatments of the desired PBD which satisfies the two conditions:

i). $3 \leq k \leq 4$;
ii). n = p and

Then pair the treatments to form the blocks of the design into three and four sizes, we shall have the following blocks:
{1, 2, 3, 4},
{1, 5, 6, 7},
{1, 8, 9, 10};
{2, 5, 8},
{2, 6, 9},
{2, 7, 10},
{3, 5, 10},
{3, 6, 8},
{3, 7, 9},
{4, 5, 9},
{4, 6, 10},
{4, 7, 8}.
Where,
Ɓ = {{1, 2, 3, 4}, {1, 5, 6, 7}, {1, 8, 9, 10}, {2, 5, 8}, {2, 6, 9}, {2, 7, 10}, {3, 5, 10}, {3, 6, 8}, {3, 7, 9}, {4, 5, 9}, {4, 6, 10}, {4, 7, 8}}.
Ɓ = $\{B_I, B_2\ldots, B_b\}$
K = $\{k_1, k_2\ldots, kb\}$
and
b = $\sum_i bi$
where, $k_1 = 3$, $k_2 = 4$, and $b_1 = 9$, $b_2 = 3$.

3.5. Confirmation of construction 1

(a). Using Connecting equation of Smith et al ([SBT98]):

$\lambda n(n-1)= \sum_i biki(ki-1)$

where, k1 = 3, k2 = 4, b1 = 9, b2 = 3, n = 10, $\lambda = 1$
Therefore,

1*10(10-1) = 9 *3 (3 – 1) + 3 * 4 (4 -1)
10(9) = 54 + 36
90 = 90
Thus,
$\lambda n(n-1)= \sum_i biki(ki-1)$

(b). Stanton-Kalbfleisch-Bound ([Wil75]):

$b \geq SK(k, n)= 1+ k^{2}\frac{(n-k)}{n-1}$

Where,
b=$\sum_i bi$ = 12, k1 = 3, k2 = 4, b1 = 9, b2 = 3, n = 10, $\lambda = 1$
Therefore,

$12 \geq SK(3, 10)$
= 1 + 9(10 -3) / 10 – 1
= 1 + 9(7) / 9
= 1+ 7
=8 , hence,
12 ≥ 8 also, when k = 4, and n = 10
$12 \geq SK(4, 10)$
= 1 + 16(6) / 9
= 1 + 10.7
= 1 + 10.7
=11. 7 hence,
$12 \geq 11.7$

This confirm construction 1.

3.6. Construction Two

For PBDs with a set block K = {3, 4, 5}

Preliminary Theorem 2. *There exists a PBD(n, {3, 4, 5}) of dimension three if and only if* $n \equiv 2, 3$(mod 4) *provided* $n \geq 11$ and $\lambda = 1$.

Here r = 4 and $\lambda = 1$.

These are used on the Li program to generate LDs that could qualify as PBDs (n, K, λ) where K = {3, 4, 5}.

Thus, the following LDs:

n	k	p	t
6	3	6	3
7	3	7	3
8	3	8	3
9	3	9	3
10	3	10	3
10	4	10	4
11	3	11	4
11	4	11	4
11	5	11	5
12	5	12	5
13	5	13	5
14	3	14	3
14	4	14	4
14	5	14	5
15	3	15	3
15	4	15	4
15	5	15	5

Conditions imposed

Several LDs were produced from using $\lambda = 1$ and r = 4 obtained from the classes of PBDs(n, {3, 4, 5}) on the Li inequality as shown above, this shows that some LDs qualified as PBDs satisfying the proposed theorem 2.

Thus, the conditions:

i). $3 \leq k \leq 5$ and

ii). n = p

which give the following LDs:

The LDs(11, 3, 11, 3), (11, 4, 11, 4), (11, 5, 11, 5)
The LDs(14, 3, 14, 3), (14, 4, 14, 4), (14, 5, 14, 5)
The LDs(15, 3, 15, 3), (15, 4, 15, 4), (15, 5, 15, 5)
The LDs (18, 3, 18, 3), (18, 4, 18, 4), (18, 5, 18, 5)
The LDs(19, 3, 19, 3), (19, 4, 19, 4), (19, 5, 22, 5)
The LDs(22, 3, 22, 3), (22, 4, 22, 4), (22, 5, 14, 5)
The LDs(23, 3,23, 5, 3), (23, 4, 23, 4), (23, 5, 23, 5)
The LDs (26, 3, 26, 3), (26, 4, 26, 4), (26, 5, 26, 5)
The LDs(27, 3, 27, 3), (27, 4, 27, 4), (27, 5, 27, 5)
The LDs(30, 3, 30, 3), (30, 4, 30, 4), (30, 5, 30, 5)
The LDs(31, 3, 31, 3), (31, 4, 31, 4), (31, 5, 31, 5)
etc.

Clearly, 3-LD(*n*, 3, p, 3), (*n*, 4, p, 4), (*n*, 5, p, 5) could be used to construct PBD(*n*, {3, 4, 5}, 1).

Theorem 2: A PBD(*n*, {3, 4, 5}) is a 3-LDs(*n*, k, p, t), (*n*, k+1, p, t), (*n*, k+2, p, t) if and only if the following conditions are satisfied:

i). $n \equiv 2, 3$ (mod 4)

ii). $3 \leq k \leq 5$

iiii). n = p and

iv). $\lambda = 1$

Derived Steps for construction of PBD(*n*, {3, 4, 5}, 1) from any 3- LDs(*n*, 3, p, 3) (*n*, 4, p, 4) (n, 5, p, 5).

(1). Select any n-treatments that satisfy the condition $n \equiv 2, 3$ (mod 4).

(2). Select any 3-LDs corresponding to n-treatments of the desired PBD that satisfy the followings conditions:

i). $3 \leq k \leq 5$;

ii). n = p and

(3). Then, generate the blocks of the PBDs by Partitioning n- treatments into sets of blocks of 3 or 4 or 5 such that each pair of treatments is contained in precisely one block which follows that $n(n-1)$ is an integer linear combination of $k(k-1)$, $k \in K$.

For Example

To construct a PBD(15, {3, 4, 5}, 1), where, n-treatment is 15 and it satisfy condition 1 thus:

X = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15}

Select

3- LDs(15, 3, 15, 3), (15, 4, 15, 4), (15, 5, 15, 5) which corresponds to n-treatment 15 and satisfy the following conditions:

i). $3 \leq k \leq 5$;

ii). n = p and

then pair the n-treatments to form the blocks of sizes three, four and five in such a way that no pair appears together more than once within blocks. Then we have the following blocks:

{1, 2, 3, 4, 5},
{1, 6, 7, 8, 9},
{1, 10, 11, 12, 13};
{2, 6, 10, 14},
{2, 7, 11, 15},
{3, 6, 12, 15},
{3, 7, 13, 14},
{4, 8, 10, 15},
{4, 9, 11, 14},
{5, 8, 12, 14},
{5, 9, 13, 15};
{1, 14, 15},
{2, 8, 13},
{2, 9, 12},
{3, 8, 11},
{3, 9, 10},
{4, 6, 13},
{4, 7, 12},
{5, 6, 11},
{5, 7, 10}.

Therefore,

$\mathcal{B} = \{B_1, B_2\ldots, B_k\}$

$K = \{k_1, k_2\ldots, k_b\}$ and

$b = \sum_i bi$

Therefore,
$k_1 = 3$, $k_2 = 4$, $k_3 = 5$, and
$b_1 = 3$, $b_2 = 8$, $b_3 = 9$
therefore b = 20.

3.7. Confirmation of construction 2

(a).Using Connecting equation of Smith et al ([SBT98]):

$\lambda n(n-1) = \sum_i biki(ki-1)$

where, k1 = 3, k2 = 4, k3 = 5, b1 = 3, b2 = 8, b3 = 9 , n = 15, λ = 1
Therefore,
1* 15(15 -1) = 9*3 (2) + 8 * 4 (3) + 3 * 5 (4)
210 = 54 + 96 +60
210 = 210
Thus,

$\lambda n(n-1) = \sum_i biki(ki-1)$

(b). Stanton-Kalbfleisch-Bound ([Wil75]):

$b \geq SK(k, n) = 1 + k^2\frac{(n-k)}{n-1}$

Where, b= 20
$20 \geq SK(3, 15)$
$= 1 + 9(17)/ 14$
$= 11.9$.
Therefore $20 \geq 11.9$
Also, when k = 4 and n = 15
$20 \geq SK(4, 15)$
$= 1 + 16(11)/ 14$
$= 13.6$.
Therefore, $20 \geq 13.6$
Also, when k = 5 and n = 15
$20 \geq SK(5, 15)$
$= 1 + 25 (10) / 14$
$= 18.9$. Therefore,
$20 \geq 18.9$
This confirm construction 2.

3.9. Characteristics of the constructed PBDs from LDs

(i) $K = \{k_1, k_2, k_3\}$
(ii) $k_i \leq n$; $k_i \neq k_j$; and
(iii) $b = \sum_i bi$

3.9. Proof of the derived theorem 1

From the Wilson Existence condition for a PBD(n, K) in (1) and (2).
Setting K = {3, 4}.
Then,

α (K) = gcd{2, 3} = 1
and
β (K) = gcd{6, 12} = 3
therefore,

n-1 ≡ 0 (mod 1) and n(n-1) ≡ 0 (mod 3).

The first condition merely says that n is an integer. The second condition is satisfied if only if

n(n -1) ≡ 0 or 1 mod 3.

Therefore, all values of n ≥ 6 such as 6, 7, 9, 10, 12, 13 could have a set of block K = {3, 4}. Then, n ≥ 6 follows since 3 is the smallest block size which corresponds to the proposed theorem.

3.10. Proof of the derived theorem 2

Setting K = {3, 4, 5}
Then,

α (K) = gcd{2, 3, 4} = 1
and
β (K) = gcd{6, 12, 20} = 4.

Therefore,

n-1 ≡ 0 (mod 1) and n(n-1) ≡ 0 (mod 4)

The first condition merely says that n is an integer. The second condition is satisfied if only if

n(n -1) ≡ 2 or 3 mod 4.

Therefore, for all n treatments equal or greater than 11 such as 11, 14, 15, 18, 19, 22, 23, 26, 27, 30, 31…, satisfying these conditions could take set of K = {3, 4, 5}. Then, n ≥ 11 follows since 3 is the smallest block size which corresponds to the proposed theorem.

4. CONCLUSION

Considering the important of Pairwise Balanced Designs (PBDs) in designs theory, the research work concluded as follows: Any LD(n, k, p, t) satisfying the Li inequality, k = 3, 4, 5 and n = p qualified as PBDs, 2-LDs(n, 3, p, 3), (n, 4, p, 4) can be used to construct PBD(n, {3, 4}) provided n ≡ 0,1(mod 3) and 3-LDs(n, 3, p, 3), (n, 4, p, 4), (n, 5, p, 5) can be used to construct PBD(n, {3, 4, 5}) provided n ≡ 2, 3 (mod 4) by Partitioning n-treatments into sets of blocks of 3 or 4 or 5 such that each pair of treatments is contained in precisely one block which follows that n(n − 1) is an integer linear combination of k(k − 1), k ∈ K.

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