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LAPLACE HOMOTOPY PERTURBATION METHOD OF SOLVING NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS

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ABSTRACT: In this paper, the Laplace Homotopy Perturbation Method (LHPM) was utilized to solve some nonlinear partial differential equations. This technique is the mixture of the Laplace transform method and the homotopy perturbation method. Some nonlinear partial differential problems of first and second order were considered which the provide results in terms of transformed variables and the series solution was obtained by applying the inverse properties of the Laplace transform. In this paper, we compared the result obtained with the available Homotopy perturbation and Elzaki transform method solution which was found to be exactly same. The results revealed that the mixture of Laplace transform and Homotopy perturbation method was quite applicable, practically well appropriate for use in such problems.

KEYWORDS: Laplace transform, Homotopy, Homotopy Perturbation method, nonlinear, partial differential equations.

1. INTRODUCTION

Diverse problems of physical interest are described by both linear and nonlinear partial differential equations with initial and boundary conditions. Linear and nonlinear partial differential equations are of fundamentally significant in science and engineering. The significance of obtaining the exact solution or approximate solutions of linear and nonlinear partial differential equations in physics and mathematics is still a significant problem that requires new techniques to obtain the exact or approximate solutions. Most of the linear and nonlinear equations do not have a definite analytic solution, thus, numerical methods have greatly been used to solve these equations. Some of the classical analytical method of solving nonlinear equations are Lyapunov artificial small parameter method ([Ado 94]), perturbation method ([BPG10, Chun09]) and Hirota bilinear method ([He99a, He99b]). In recent years, several researchers have paid attention to studying the solutions partial differential equations using diverse techniques. Some of these are Adomian decomposition method ([Ado07]), the homotopy analysis method ([I+10, J+09, Lia92]), the homotopy perturbation method ([SNG09, He06]). Majority of all these methods have their shortcomings such the calculations of Adomians polynomial have divergent results and huge computational work.

Some integral transformation techniques include Elzaki ([EH12, Elz11]), Laplace ([HM10, KG09]), Sumudu ([EK10a, Zha07]), Aboodh ([Abo14, Abo13]) have been applied to solve linear partial differential equations. The integral transform has the capacity of transforming differential equations into algebraic equations which give room for systematic approach.

But using integral transform in solving nonlinear differential equations may increase the complexity. Recently, several researchers have obtained solutions of nonlinear differential equations by using diverse approach. In this paper, we use coupling of Laplace transform and homotopy perturbation method. This technique is a useful technique for solving both linear and nonlinear differential equations. The major aim of this paper is to consider the effectiveness of the Laplace homotopy perturbation method in solving nonlinear partial differential equation both homogeneous and non-homogeneous.

2. LAPLACE HOMOTOPY PERTURBATION METHOD

To illustrate the fundamental thought of this method, we consider a general type of two-dimensional nonlinear second order non-homogeneous partial differential equations with variable coefficients of the form

$$\frac{d^2u(x,t)}{dt^2} + Ru(x,t) + Nu(x,t) = g(x,t) \qquad (2.1)$$

where $\frac{d^2u(x,t)}{dt^2}$ is the linear differential operator of order 2, R is a linear operator , N is a nonlinear operator and $g(x,t)$ is the source function.

The initial and boundary conditions associated with the problem is of the form $u(x,0) = h(x)$

$$u_t(x,0) = f(x), t > 0 \quad (2.2)$$

By applying the Laplace transform and the use of the linearity property of Laplace transform

$$L\left\{\frac{d^2u(x,t)}{dt^2}\right\} + L\{Ru(x,t)\} + L\{Nu(x,t)\} = L\{g(x,t)\} \quad (2.3)$$

$$s^2u(x,s) - su(x,0) - u_t(x,0) + L\{Ru(x,t) + Nu(x,t)\} = L\{g(x,t)\} \quad (2.4)$$

Applying the initial condition in equation (2.2) into (2.4).

$$s^2u(x,s) - sh(x) - f(x) + L\{Ru(x,t) + Nu(x,t)\} = L\{g(x,t)\}$$

$$u(x,s) = \frac{1}{s}h(x) + \frac{1}{s^2}f(x) - \frac{1}{s^2}L\{Ru(x,t) + Nu(x,t)\} + \frac{1}{s^2}L\{g(x,t)\} \quad (2.5)$$

Taking the Laplace inverse of equation (2.5).

$$U(x,t) = G(x,t) - L^{-1}\left\{\frac{1}{s^2}L\{Ru(x,t) + Nu(x,t)\}\right\}$$

$G(x,t)$ represents the term occurring from the source function $g(x,t)$ and the associated initial conditions.
Now, we apply the Homotopy perturbation method

$$u(x,t) = \sum_{n=0}^{\infty} P^n u_n(x,t) \quad (2.6)$$

The nonlinear term can be decomposed in the following way:

$$N[u(x,t)] = \sum_{n=0}^{\infty} P^n u_n(x,t) \quad (2.7)$$

Using the He's polynomial H_n(u) ([EH12]) given as follows:

$$H_n(u_0, \cdots u_n) = \frac{1}{\Gamma(n+1)}\frac{\partial^n}{\partial p^n}\left[N\left(\sum_{n=0}^{\infty} P^n u_n(x,t)\right)\right], n = 0,1,2,\cdots \quad (2.8)$$

Substituting equation (2.7) and (2.8) into (2.6), we obtain

$$\sum_{n=0}^{\infty} P^n u_n(x,t) = G(x,t) - P\left[L^{-1}\left\{\frac{1}{s^2}L\left\{R\sum_{n=0}^{\infty} P^n u_n(x,t) + \sum_{n=0}^{\infty} P^n H_n(x,t)\right\}\right\}\right] \quad (2.9)$$

This is the coupling of the Laplace transform and the homotopy perturbation method using He's polynomials.
Comparing the coefficients of the like power of p, the following approximations are obtained:

$$\begin{aligned}
P^0&: u_0(x,t) = G(x,t)\\
P^1&: u_1(x,t) = -L^{-1}\left\{\frac{1}{s^2}L\{Ru_0(x,t) + H_0(u)\}\right\}\\
P^2&: u_2(x,t) = -L^{-1}\left\{\frac{1}{s^2}L\{Ru_1(x,t) + H_1(u)\}\right\}\\
P^3&: u_3(x,t) = -L^{-1}\left\{\frac{1}{s^2}L\{Ru_2(x,t) + H_2(u)\}\right\}\\
P^4&: u_4(x,t) = -L^{-1}\left\{\frac{1}{s^2}L\{Ru_3(x,t) + H_3(u)\}\right\}\\
&\vdots\\
P^n&: u_n(x,t) = -L^{-1}\left\{\frac{1}{s^2}L\{Ru_{n-1}(x,t) + H_{n-1}(u)\}\right\}
\end{aligned} \quad (2.10)$$

The approximate analytical solution of equation (2.1) is written in truncated series as

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + u_3(x,t) + \cdots \quad (2.11)$$

Equation (2.11) can be written as

$$u(x,t) = u_0(x,t) + \sum_{n=1}^{\infty} u_n(x,t) \quad (2.12)$$

3. APPLICATION

In order to demonstrate the effectiveness of Laplace Homotopy Perturbation method for solving nonlinear partial differential equations, we will consider the following three examples.

Example 1:

Let us first consider first order homogeneous nonlinear partial differential equation as ([EH12])

$$u_t + uu_x = 0, \quad (3.1)$$
$$u(x,0) = -x \quad (3.2)$$

Applying Laplace transform to equation (3.1), admits to

$$L\{u_t + uu_x\} = 0$$
$$L\{u_t\} + L\{uu_x\} = 0 \quad (3.3)$$
$$su(x,s) - u(x,0) + L\{uu_x\} = 0$$

Using the initial conditions from equation (3.2), we have

$$su(x,s) + x + L\{uu_x\} = 0$$
$$u(x,s) = -\frac{1}{s}x - \frac{1}{s}L\{uu_x\} \quad (3.4)$$

Taking the Laplace inverse of both sides of equation (3.4)

$$u(x,t) = -x - L^{-1}\left\{\frac{1}{s}L\{uu_x\}\right\} \quad (3.5)$$

Now, we apply the homotopy perturbation method.

$$u(x,t) = \sum_{n=0}^{\infty} P^n u_n(x,t) \quad (3.6)$$

$$\sum_{n=0}^{\infty} P^n u_n(x,t) = -x - PL^{-1}\left\{\frac{1}{s}L\left\{\sum_{n=0}^{\infty} P^n H_n(u)\right\}\right\} \quad (3.7)$$

$H_n(u)$ are He's polynomials that stands for the nonlinear terms.

The first few components of He's polynomials are given by

$$H_0(u) = u_0 u_{0x}$$
$$H_1(u) = u_0 u_{1x} + u_1 u_{0x} \quad (3.8)$$
$$H_2(u) = u_0 u_{2x} + u_1 u_{1x} + u_2 u_{0x}$$

Comparing the coefficients of like powers of p in equation (3.7) the following approximations are obtained:

$$P^0: \ u_0(x,t) = -x, \qquad H_0(u) = x$$
$$P^1: \ u_1(x,t) = -L^{-1}\left\{\frac{1}{s}L\{H_0(u)\}\right\}$$
$$P^1: \ u_1(x,t) = -L^{-1}\left\{\frac{1}{s}L\{x\}\right\}$$
$$u_1(x,t) = -L^{-1}\left\{\frac{1}{s^2}\right\}$$
$$u_1(x,t) = -xt$$
$$H_1(u) = u_0 u_{1x} + u_1 u_{0x}$$
$$H_1(u) = -x(-t) + (-xt)(-1)$$
$$H_1(u) = 2xt$$
$$P^2: \ u_2(x,t) = -L^{-1}\left\{\frac{1}{s}L\{H_1(u)\}\right\}$$
$$P^2: \ u_2(x,t) = -L^{-1}\left\{\frac{1}{s}L\{H_1(u)\}\right\}$$
$$P^2: \ u_2(x,t) = -L^{-1}\left\{\frac{1}{s}L\{2xt\}\right\}$$
$$u_2(x,t) = -L^{-1}\left\{\frac{1}{s}\left(\frac{2x}{s^2}\right)\right\}$$
$$u_2(x,t) = -L^{-1}\left\{\frac{2x}{s^3}\right\}$$
$$u_2(x,t) = -xt^2$$
$$H_2(u) = u_0 u_{2x} + u_1 u_{1x} + u_2 u_{0x}$$
$$H_2(u) = (-x)(-t^2) + (-xt)(-t) + (-xt^2)(-1)$$
$$= xt^2 + xt^2 + xt^2$$
$$= 3xt^2$$
$$P^3: \ u_3(x,t) = -L^{-1}\left\{\frac{1}{s}L\{H_2(u)\}\right\}$$
$$P^3: \ u_3(x,t) = -L^{-1}\left\{\frac{1}{s}L\{3xt^2\}\right\}$$
$$u_3(x,t) = -L^{-1}\left\{\frac{1}{s}\left(\frac{6x}{s^3}\right)\right\}$$
$$u_3(x,t) = -L^{-1}\left\{\frac{6x}{s^4}\right\}$$
$$u_3(x,t) = -xt^3$$

Following the same pattern, the rest of the components $u_n(x,t)$ can be found.

In general, the recursive relation is given by:

$$P^m: \ u_m(x,t) = -L^{-1}\left\{\frac{1}{s}L\{H_{m-1}(u)\}\right\} \quad (3.9)$$

Therefore, the solution $u(x,t)$ of equation (3.1) is given by

$$u(x,t) = -x - xt - xt^2 - xt^3 - \cdots$$
$$u(x,t) = -x(t + t^2 + t^3 + \cdots) \quad (3.10)$$

in series form, and $u(x,t) = \frac{x}{t-1}$ is in closed form, which is the same as the result in ([EH12]).

Example2:

Consider the first order nonlinear partial differential equation ([EH12])

$$u_t + uu_x = 2t + x + t^3 + xt^2, u(x,0) = -x \quad (3.11)$$

Applying the Laplace transform to equation (3.11) subject to the initial condition

$$\begin{aligned}
&L\{u_t\}+L\{uu_x\}=L\{2t+x+t^3+xt^2\}\\
&L\{u_t\}+L\{uu_x\}=L\{2t\}+L\{x\}+L\{t^3\}+L\{xt^2\}\\
&su(x,s)+x+L\{uu_x\}=\frac{2}{s^2}+\frac{x}{s}+\frac{2x}{s^3}+\frac{6}{s^4}\\
&u(x,s)=\frac{2}{s^3}+\frac{x}{s^2}+\frac{2x}{s^4}+\frac{6}{s^5}-\frac{x}{s}-\frac{1}{s}L\{uu_x\}
\end{aligned}\tag{3.12}$$

Now taking the inverse Laplace on both sides of equation (3.12)

$$u(x,t)=t^2+xt+\frac{xt^3}{3}+\frac{t^4}{4}-x-L^{-1}\left\{\frac{1}{s}L\{uu_x\}\right\}\tag{3.13}$$

Applying the Homotopy Perturbation Method, we obtain:

$$\begin{aligned}
&\sum_{n=0}^{\infty}P^n\,u_n(x,t)=t^2+xt+\frac{xt^3}{3}+\frac{t^4}{4}-x-PL^{-1}\left\{\frac{1}{s}L\{\textstyle\sum_{n=0}^{\infty}P^nH_n(u)\}\right\}\\
&H_0(u)=u_0u_{0x}\\
&H_1(u)=u_0u_{1x}+u_1u_{0x}\\
&H_2(u)=u_0u_{2x}+u_1u_{1x}+u_2u_{0x}
\end{aligned}\tag{3.14}$$

Comparing the coefficients of the like powers of P. The following components of the series solution are obtained:

$$P^0: u_0(x,t)=t^2-x+xt+\frac{xt^3}{3}+\frac{t^4}{4}\tag{3.15}$$

$$\begin{aligned}
&H_0(u)=u_0u_{0x}\\
&H_0(u)=\left(t^2-x+xt+\frac{xt^3}{3}+\frac{t^4}{4}\right)\left(-1+t+\frac{t^3}{3}\right)\\
&H_0(u)=-t^2+t^3+\frac{t^5}{3}+x-2xt-\frac{xt^3}{3}+xt^2+\frac{xt^4}{3}-\frac{t^4}{4}+\frac{t^5}{4}+\frac{t^7}{12}\\
&u_1(x,t)=-L^{-1}\left\{\frac{1}{s}L\{H_0(u)\}\right\}\\
&P^1: u_1(x,t)=-\frac{t^4}{4}-\frac{xt^3}{5}-\frac{xt^5}{15}-\frac{xt^7}{63}-\frac{7}{72}t^6-\frac{t^8}{96}\\
&\quad\vdots\qquad\vdots\qquad\vdots
\end{aligned}\tag{3.16}$$

The noise terms appear between the components $u_0(x,t), u_1(x,t)$
Thus, the exact solution is

$$u(x,t)=t^2+xt\tag{3.17}$$

which is the same as the result in ([EH12]).

Example 3:
Consider the second order nonlinear partial differential equation ([EH12])

$$\frac{\partial u}{\partial t}=\left(\frac{\partial u}{\partial x}\right)^2+u\frac{\partial^2 u}{\partial x^2},\qquad u(x,0)=x^2\tag{3.18}$$

Applying the Laplace transform to equation (3.18) subject to the initial condition, we have

$$L\left\{\frac{\partial u}{\partial t}\right\}=L\left(\frac{\partial u}{\partial x}\right)^2+L\left\{u\frac{\partial^2 u}{\partial x^2}\right\}\tag{3.19}$$

$$\begin{aligned}
&su(x,s)-u(x,0)=L\left\{\left(\frac{\partial u}{\partial x}\right)^2+u\frac{\partial^2 u}{\partial x^2}\right\}\\
&su(x,s)-x^2=L\left\{\left(\frac{\partial u}{\partial x}\right)^2+u\frac{\partial^2 u}{\partial x^2}\right\}\\
&u(x,s)=\frac{1}{s}x^2+\frac{1}{s}L\left\{\left(\frac{\partial u}{\partial x}\right)^2+u\frac{\partial^2 u}{\partial x^2}\right\}
\end{aligned}\tag{3.20}$$

Employing the inverse Laplace transform on both sides of equation (3.20) we have

$$u(x,t)=x^2+L^{-1}\left\{\frac{1}{s}L\left\{\left(\frac{\partial u}{\partial x}\right)^2+u\frac{\partial^2 u}{\partial x^2}\right\}\right\}\tag{3.21}$$

Now, applying the homotopy perturbation method we have,

$$\begin{aligned}
&\sum_{n=0}^{\infty}P^n\,u_n(x,t)=x^2+PL^{-1}\left\{\frac{1}{s}L\{\textstyle\sum_{n=0}^{\infty}P^nH_n(u)\}\right\}\\
&H_0(u)={u^2}_{0x}+\ u_0u_{0xx}\\
&H_1(u)=2u_{0x}u_{1x}+u_0u_{1xx}\ +u_1u_{0xx}
\end{aligned}\tag{3.22}$$

Comparing the coefficients of the like powers of P. The following components of the series solution are obtained:

$$\begin{aligned}
&P^0:\ u_0(x,t)=x^2\\
&H_0(u)=6x^2\\
&P^1:\ u_1(x,t)=L^{-1}\left\{\frac{1}{s}L\{H_0(u)\}\right\}\\
&=L^{-1}\left\{\frac{1}{s}L\{6x^2\}\right\}\\
&=L^{-1}\left\{\frac{1}{s}\left(\frac{6x^2}{s}\right)\right\}\\
&u_1(x,t)=6x^2t\\
&H_1(u)=2u_{0x}u_{1x}+u_0u_{1xx}\ +u_1u_{0xx}\\
&u_0=x^2,\qquad u_{0x}=2x,\qquad u_{1x}=12xt,\\
&\qquad u_{0xx}=2,\qquad u_{1xx}=12t\\
&H_1(u)=2(2x)(12xt)+x^2\ +(6x^2t)2=72x^2t
\end{aligned}$$

$$P^2:\ u_2(x,t) = L^{-1}\left\{\frac{1}{s}L\{H_1(u)\}\right\}$$
$$= L^{-1}\left\{\frac{1}{s}L\{72x^2t\}\right\}$$
$$= L^{-1}\left\{\frac{1}{s}\left(\frac{72x^2}{s^2}\right)\right\}$$
$$= L^{-1}\left\{\frac{72x^2}{s^3}\right\}$$
$$= 36x^2t^2$$

The rest of the components of the iteration can be obtained by following the same approach.
Thus, the solution $u(x,t)$ is given by:

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + \cdots$$
$$u(x,t) = x^2 + 6x^2t + 36x^2t^2$$

Therefore, the solution $u(x,t)$ is given by in series form, and

$$u(x,t) = x^2(1 + 6t + 36t^2)$$

is in closed form

$$u(x,t) = \frac{x^2}{1-6t} \qquad (3.23)$$

which is the same as the result in ([EH12])

4. CONCLUSIONS

The fundamental objective of this paper is to demonstrate the applicability of the mixture of Laplace transform and homotopy perturbation method to solve homogeneous and nonhomogeneous nonlinear partial differential equations. The mixture of the two methods successfully worked to give a very reliable and exact solutions to the equation. This technique provides an analytical approximation in a rapidly convergent sequence with elegantly computed terms. This method solves nonlinear partial differential equations without applying Adomian's polynomials. Subsequently, this technique can be applied to other nonlinear partial differential equations.

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