ROBUST THREE-STEP BROYDEN – LIKE ALGORITHMS FOR FUNCTIONS OF SEVERAL VARIABLES

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ABSTRACT: In this work, we suggest some variants of Broyden-like algorithm to solve systems of nonlinear equations using various combination of quadrature rules . The proposed methods are three-step in nature whereby the first step is the initial iterate of the Newton's method and the remaining two steps are the predictor-corrector of Broyden method and quadrature rules. Numerical tests show that these methods give encouraging results with respect to some existing Broyden-like methods in view of Dolan and More performance profile and Bogle and Perkins comparison index.
KEYWORDS: Broyden-like method, quadrature formulas, predictor-corrector model, systems of nonlinear equations, numerical examples.

1. INTRODUCTION

Consider the system of nonlinear equations

$$F(x) = 0 \tag{1}$$

where $F : D \subset \Re^n \to \Re^n$ is a continuously differentiable function. In practice, algorithms for solving (1) are iterative in nature as solutions to such equations are rather not in existence or difficult to find using analytical methods. Classical Newton's method is the prominent method used in applications due to its convergence property. Different Newton's like schemes have been proposed in the literature in order to avert challenges associated with classical Newton's method with regard to solution quality and computational effort [MW15, MMW12]. Among them is the fixed Newton method which avoids the computation of the Jacobian but challenged with slower convergence rate [WLM12]. Dembo et al. [DES12] proposed an inexact Newton method which computes an approximate solution to the Newton equation. Other numerical methods similar to Newton-like schemes proposed in literature adopted some techniques such as matrix-free secant method, quadrature rules, high order Newton's method and composition scheme; see for example, [FS03, SDK10, DS11, WF00, WLM12, CT06, CT07, SAN08, FS04 and CTV12]. In the recently proposed Newton-like model of Saheya et al. [S+16], a new rational approximation technique is introduced for system of nonlinear equations. The novel method revises the Jacobian matrix by a rank one matrix each iteration and obtains quadratic convergence property.

Another efficient approach that approximates the Jacobian or its inverse with less computational matrix is the quasi- Newton method. But despite the emergence of Broyden's method as one of the famous and simplest quasi-Newton; the scheme still requires more number of iterations to converge to a solution.

In an attempt to reduce the number of iterations required by the classical Broyden to converge to a solution, various Broyden-like methods have been introduced. These attempts are mainly because of the influences and applications of nonlinear equations of the form (1) in science, engineering and chemical problems.

Authors such as [MMW14, MW15, OY17, OY18] and references therein) proposed quadrature-based Broyden-like methods to solve nonlinear system of equations. In Mamat et al. [MMW14], authors proposed a Broyden-like method using trapezoidal rule while the authors Muhammed and Waziri [MW15] employed the weight combination of the midpoint and Simpson quadrature formulas. Osinuga and Yusuff [OY17, OY18] employed the use of appropriate combination of trapezoidal and Simpson as well as trapezoidal, midpoint and Simpson quadrature rules respectively. In each case, authors provided better estimations of the integral from the fundamental theorem of calculus using quadrature formulas and then replacing the Jacobian obtained with Broyden's matrix. In addition, the initial Jacobian matrix was taken as an identity matrix and their approaches adopted the predictor-corrector techniques where the classical Broyden is the predictor and the proposed method is the corrector.

Muhammed et al. [MMW13] have proposed Broyden-like hybrid scheme based on true Jacobian for the first iteration and Broyden's update for subsequent iterations for the solution of nonlinear equations. The scheme was shown to be encouraging and convergence property established under mild conditions.

In this paper, we are interested in a robust and competitive scheme for the solution of (1). Motivated by the idea in Muhammed at al. [MMW13], we propose a three point predictor-corrector model. Unlike Osinuga and Yusuff [OY17, OY18], [MMW14] and [MW15], our study employ predictor-corrector approach (after true Jacobian for initial iterations) where the classical Broyden is the predictor in the second step and proposed variants based on quadrature formulas is the corrector in the third step. This method reduces the number of iterations required to converge to a solution as well as function evaluation. The rest of the paper is organized as follows. We develop in the next section variants of Newton-like schemes. Some numerical experiment results are reported in section 3 while section 4 is for brief conclusion.

2. DESCRIPTION OF THREE-STEP VARIANTS OF BROYDEN-LIKE METHOD

In this section we present the proposed schemes, the idea are to start by evaluating the Jacobian at x_0 and finding the first iterate x_1 by Newton's method. We proceed by using the predictor-corrector approach to find the next iterates where the classical Broyden as predictor method and then use the proposed schemes as corrector method based on weight combination of quadrature formulas.

It is has been shown [SAN08] that the quadrature formulas play an important and significant role in the evaluation of the integrals from the fundamental theorem of calculus (See [SAN08, MMW14, etc.]). The quadrature formulas have been utilized to develop some iterative schemes for solving nonlinear equations in the literature (see [MMW14, MW15, SAN08, OY17, OY18, CT07, FS04] and references therein). In the same vein, we develop and analyze further some iterative methods using weight combination of trapezoidal, midpoint and Simpson quadrature formulas. Thus, we present the algorithms for variants of three-step Broyden-like methods:

Three-step midpoint-Simpson (3MS) Algorithm: Given initial guess x_0 and letting $B_0 = J(x_0)$, compute the approximation solution x_{k+1} by the iterative schemes

$$x_1 = x_0 + J_0^{-1}F(x_0)$$
$$m_k = x_k - B(x_k)^{-1}F(x_k)$$
$$x_{k+1} = x_k - 12[B(x_k) + 10B(z_k) + B(m_k)]^{-1}F(x_k)$$

for $z_k = \frac{m_k + x_k}{2}, k = 0,1,\ldots$

Three-step midpoint-trapezoidal (3MT) Algorithm: Given initial guess x_0 and letting $B_0 = J(x_0)$, compute the approximation solution x_{k+1} by the iterative schemes

$$x_1 = x_0 + J_0^{-1}F(x_0)$$
$$m_k = x_k - B(x_k)^{-1}F(x_k)$$
$$x_{k+1} = x_k - 4[B(x_k) + 2B(z_k) + B(m_k)]^{-1}F(x_k)$$

for $z_k = \frac{m_k + x_k}{2}, k = 0,1,\ldots$

Three-step trapezoidal-Simpson (3TS) Algorithm: Given initial guess x_0 and letting $B_0 = J(x_0)$, compute the approximation solution x_{k+1} by the iterative schemes

$$x_1 = x_0 + J_0^{-1}F(x_0)$$
$$m_k = x_k - B(x_k)^{-1}F(x_k)$$
$$x_{k+1} = x_k - 3[B(x_k) + B(z_k) + B(m_k)]^{-1}F(x_k$$

for $z_k = \frac{m_k + x_k}{2}, k = 0,1,\ldots$

Three-step trapezoidal-Simpson-midpoint (3TSM) Algorithm: Given initial guess x_0 and letting $B_0 = J(x_0)$, compute the approximation solution x_{k+1} by the iterative schemes

$$x_1 = x_0 + J_0^{-1}F(x_0)$$
$$m_k = x_k - B(x_k)^{-1}F(x_k)$$
$$x_{k+1} = x_k - 24[5B(x_k) + 14B(z_k) + 5B(m_k)]^{-1}F(x_k)$$

for $z_k = \frac{m_k + x_k}{2}, k = 0,1,\ldots$

3. NUMERICAL RESULTS

This section presents the numerical results and discussion of the proposed schemes. We use 5 test problems from [R+90] with dimensions of 3 to 1065 variables to test the performances of the proposed schemes against classical Broyden (CB) [Bro65], Broyden-like hybrid method (BL) [MMW13], midpoint- Simpson method (MSB) [MW15], trapezoidal-Simpson-midpoint method (TSMM) [OY17], and trapezoidal-Simpson (TS) [OY18]. In the following, we give details of the used test functions:

Problem 1 [R+90]:

$$(3-\gamma x_i)+1-x_{i-1}-2x_{i+1}\,,$$
$$x_0=x_{n+1}=0\,,\gamma=0.1\,,$$
$$x_0=(-1,\dots,-1)^T\,,i=1,\dots,n-1$$

Problem 2 [R+90]:

$$(3-2x_i)x_i+1-x_{i-1}-2x_{i+1}\,,$$
$$x_0=x_{n+1}=0\,,$$
$$x_0=(-1,\dots,-1)^T\,,i=1,\dots,n-1$$

Problem 3 [R+90]:

$$x_{i-1}-2x_i+x_{i+1}-h^2\exp(x_i)\,,i=1,\dots,n-1$$
$$x_0=x_{n+1}=0\,,$$
$$h={}^1/{}_{n}+1\,,$$
$$x_0=(0,\dots,0)^T$$

Problem 4 [R+90]:

$$-x_{i-1}+2x_i-x_{i+1}+h^2(x_i+\sin x_i),$$
$$i=1,\dots,n-1$$
$$x_0=x_{n+1}=1\,,$$
$$h={}^1/{}_{n}+1\,,$$
$$x_0=(1,\dots,1)^T$$

Problem 5 [R+90]:

$$-x_{i-1}+2x_i-x_{i+1}+h^2(x_i+\sin x_i),$$
$$i=1,\dots,n-1$$
$$x_0=0,x_{n+1}=1\,,$$
$$h={}^1/{}_{n}+1\,,$$
$$x_0=(1,\dots,1)^T$$

In order to evaluate the performance of the suggested schemes, the comparison was carried out based on the number of iterations and Robustness index using performance profile of [DM02] and the comparison index proposed by [BP90] respectively.
The computational experiments were carried out on MATLAB 2017a with a double precision arithmetic. In all our examples, the maximum number of iteration is *n=500* and we used a stopping criterion $\|F(x_k)\|\leq 10^{-14}$ for the computer programs if no x satisfies. A failure is reported (denoted by '-') in the tabulated result.
Table 1 bellow give numerical result of the five test functions used in the experiments. It can be clearly seen that the proposed schemes showed a significant improvement in the number of iterations performed as compared to CB, BL, TS, MSB and TSMM. The proposed schemes solved all the tested problems along with the BL but still showed its superiority as the dimension of variables increases by converging with lesser number of iterations.
On the set of benchmark problems in Table 2, this profile uses Robustness index. The Table shows that in terms of Robustness, the proposed schemes are superior only to CB, MSB, TS and TSMM methods.
In Figure 1, 3MSB, 3TSMM, 3MT, 3TS and BL climb off the graph and this indicates that they solved all the tested problems successfully. Besides, we can say that the 3MSB is the fastest solver follow by 3TSMM, 3MT, 3TS and BL based on the number of iterations pertaining to Figure 1.

4. CONCLUSION

Based on the numerical results obtained with the tested problems, robust variants of Broyden method have been proposed. The suggested algorithms give encouraging results and can be used in place of the existing Broyden-like methods especially for large scale systems. The solution of nonlinear systems of equations using parametric Broyden-like method is an interesting area for further investigation.

Table 1: Examples and comparison between algorithms

Problem	n	CB	BL	MSB	TSMM	TS	3MSB	3TSMM	3TS	3MT
1	3	10	8	7	7	8	4	4	4	4
	35	61	22	29	30	31	11	10	12	11
	65	91	30	39	38	38	15	15	14	14
	165	-	46	47	44	43	22	22	22	22
	365	-	74	49	44	43	30	29	30	29
	665	-	107	52	45	44	38	39	36	40
	1065	-	160	-	46	45	47	45	42	40
2	3	-	10	9	13	-	7	8	8	8
	35	-	16	34	35	37	10	11	12	11
	65	-	16	46	46	46	10	11	12	11
	165	-	16	55	51	53	10	11	12	11
	365	-	16	55	52	-	10	11	12	11
	665	-	16	55	54	-	10	11	12	11
	1065	-	16	56	-	-	10	11	12	11
3	3	8	4	7	9	10	3	3	4	3
	35	107	3	72	78	-	3	3	3	3
	65	-	3	-	-	-	2	3	3	3
	165	-	3	-	-	-	2	3	3	3
	365	-	3	-	-	-	2	2	3	2
	665	-	3	-	-	-	2	2	2	2
	1065	-	3	-	-	-	2	2	2	2
4	3	7	4	4	6	6	3	4	4	4
	35	51	4	36	30	29	3	3	3	3
	65	104	4	76	71	68	3	3	3	3
	165	-	3	422	483	416	2	3	3	3
	365	-	3	-	-	-	2	3	3	3
	665	-	3	-	-	-	2	2	3	3
	1065	-	3	-	-	-	2	2	2	2
5	3	11	6	6	7	8	4	4	5	4
	35	86	5	62	59	70	4	4	4	4
	65	252	5	149	120	113	4	4	4	4
	165	-	5	-	-	417	3	4	4	4
	365	-	4	-	-	-	3	4	4	4
	665	-	4	-	-	-	3	4	4	4
	1065	-	4	-	-	-	3	3	4	4

Table 2: Robustness indices for the 5 problems

	CB	BL	MSB	TSMM	TS	3MSB	3TSMM	3TS	3MT
R	0.422	1.000	0.711	0.711	0.644	1.000	1.000	1.000	1.000

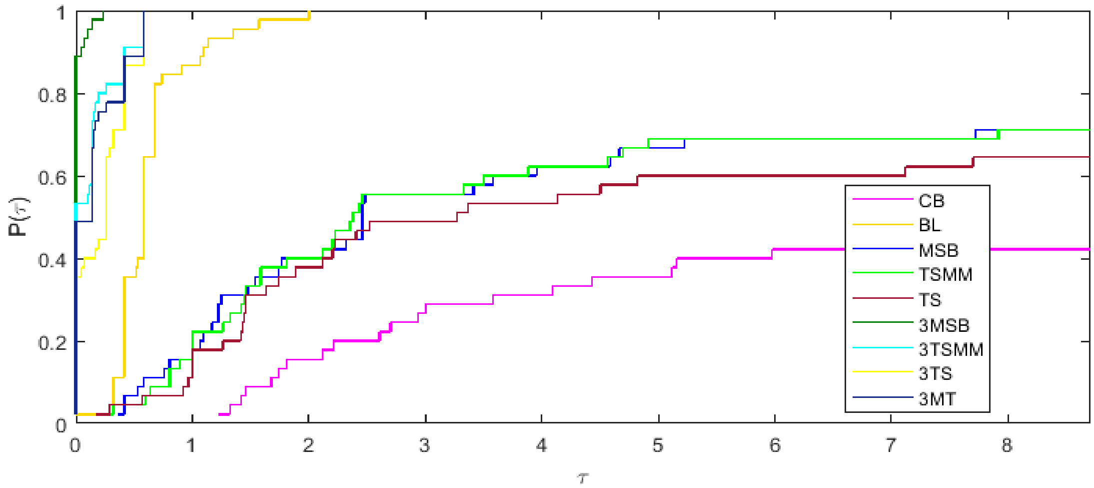

Figure 1: Performance Profile by number of iterations

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