

ON THE COMPARISON BETWEEN PICARD ITERATION METHOD AND ADOMIAN DECOMPOSITION METHOD IN SOLVING NON LINEAR DIFFERENTIAL EQUATIONS

O. Oludoun, O. Adebimpe, B. Gbadamosi, O. E. Abiodun

Department of Physical Sciences, Landmark University, Omu-Aran, Nigeria

Corresponding Author: O. Oludoun, oludoun.olajumoke@lmu.edu.ng

ABSTRACT: In this paper, investigation of two analytical series methods for solving non-linear differential equations is examined. The study is to give a comparison between Picard's iteration method and Adomian decomposition method in solving non-linear differential equations. The methods will be compared in terms of convergence, accuracy and efficiency.

KEYWORDS: Adomian decomposition method, Convergence of the Adomian method, Picard's iteration method, Picard's Existence and Uniqueness Theorem.

1. INTRODUCTION

Many differential equations cannot be solved by standard methods, in such equations, it is sufficient to obtain an approximation solution. Among the approximation method used for solving differential equations, are the Picard iteration method and the Adomian decomposition method. The Picard iteration method named after a French mathematician Charles Emile Picard (1856–1941) is a successive approximation method which is used in giving an approximate solution to an initial-value problem. It is also a constructive method used for initiating the existence and the uniqueness of a solution to a differential equation of the form $y' = f(x, y)$. The Adomian decomposition method (ADM) is an approximation method used for solving non-linear differential equation [Ado88]. This method involves the decomposition of a solution of a non-linear equation in a series of functions whereby each term of the series is obtained recursively from a polynomial generated from an expansion of an analytic function into a power series [Ado92]. These polynomials are called Adomian polynomials. ADM is most useful in non-linear differential equations and the number of the partial solutions used determines their accuracy. One of the advantages of this method is that it converges fast to the exact solution [Ado92].

2. PICARD ITERATION METHOD (PIM)

The Picard iteration method is an iteration method

which is used to provide an existence solution to initial value problems [CL55]. In this chapter, the analysis of the Picard iteration method will be shown and the detailed proof of the existence and uniqueness theorem will be given.

2.1. Analysis of the Picard Iteration Method

In this section, the Picard iteration method will be analysed as shown in [NSS04]. Consider an initial value problem:

$$y'(x) = f(x, y), \quad y(x_0) = y_0. \quad (1)$$

Equation (1) can be rewritten in an integral form:

$$\int_{x_0}^x y'(x)dx = \int_{x_0}^x f(x, y(x))dx, \\ y(x) - y(x_0) = \int_{x_0}^x f(x, y(x))dx. \quad (2)$$

Substituting the initial condition $y(x_0) = y_0$ into equation (2) we have:

$$y(x) - y_0 = \int_{x_0}^x f(x, y(x))dx, \\ y(x) = y_0 + \int_{x_0}^x f(x, y(x))dx. \quad (3)$$

Changing the variable of integration to t instead of x in (3) gives,

$$y(x) = y_0 + \int_{x_0}^x f(t, y(t))dt. \quad (4)$$

Equation (4) is used to generate successive approximation of solution to initial-value problems which is of the form:

$$y_n(x) = y_0 + \int_{x_0}^x f(t, y_{n-1}(t))dt, \\ n = 1, 2, 3, \dots \quad (5)$$

Hence the successive approximations are of the

form:

$$\begin{aligned} y_0(x) &= y_0, \\ y_1(x) &= y_0 + \int_{x_0}^x f(t, y_0(t))dt, \\ y_2(x) &= y_0 + \int_{x_0}^x f(t, y_1(t))dt, \\ &\vdots \\ y_{n+1}(x) &= y_0 + \int_{x_0}^x f(t, y_n(t))dt. \end{aligned} \quad (6)$$

Picard's Existence and Uniqueness Theorem (Cauchy Lipschitz Theorem)

In this section, we prove the Picard's existence and uniqueness theorem as presented by [Lin94].

Theorem 2.1

Let $f(x, y)$ be a continuous function of (x, y) plane in a region D . Let M be a constant such that

$$|f(x, y)| < M \text{ in } D. \quad (7)$$

Also, let $f(x, y)$ in D satisfy the Lipschitz condition:

$$|f(x, y_1) - f(x, y_2)| \leq A|y_1 - y_2|, \quad (8)$$

where A is independent of x, y_1, y_2 . Suppose there exist a rectangle R defined by

$$|x - x_0| \leq h, \quad |y - y_0| \leq k, \quad (9)$$

such that $Mh < k$ where $R \subseteq D$. Then for $|x - x_0| \leq h$, $y' = f(x, y)$ has a unique solution $y = y(x)$ for which $y(x_0) = y_0$.

Proof. Assume that x is defined such that $|x - x_0| \leq h$. Let's also define the series of functions:

$$\begin{aligned} y_0(x) &= y_0, \\ y_1(x) &= y_0 + \int_{x_0}^x f(x, y_0(x))dx, \\ y_2(x) &= y_0 + \int_{x_0}^x f(x, y_1(x))dx, \\ &\vdots \\ y_{n-1}(x) &= y_0 + \int_{x_0}^x f(x, y_{n-2}(x))dx, \\ y_n(x) &= y_0 + \int_{x_0}^x f(x, y_{n-1}(x))dx. \end{aligned} \quad (10)$$

We will then consider the proof in five main steps:

First step: To show that for $x_0 - h \leq x \leq x_0 + h$, there exists a curve $y = y_n(x)$ which lies in the rectangle R i.e. $\exists y_0 - k \leq y \leq y_0 + k$. Now, suppose $y = y_{n-1}(x)$ lies in R so that $f(x, y_{n-1}(x))$ is defined, continuous and satisfies $|f(x, y_{n-1}(x))| \leq M$ on the interval $x_0 - h \leq x \leq$

$x_0 + h$. Then from equation (10),

$$\begin{aligned} |y_n(x) - y_0(x)| &= \left| \int_{x_0}^x f(x, y_{n-1}(x))dx \right|, \\ &= \left| \int_{x_0}^x f(x, y_{n-1}(x)) \right| |dx|, \\ &\leq M|x - x_0|, \\ &\leq Mh < k. \end{aligned}$$

We have clearly shown that $y_n(x)$ lies in R , and hence $f(x, y_n(x))$ is defined and continuous on the interval $x_0 - h \leq x \leq x_0 + h$.

Second step: To show that by induction $y = y_n(x)$ lies in R so that:

$$|y_n(x) - y_{n-1}(x)| \leq \frac{MA^{n-1}}{n!} |x - x_0|^n. \quad (11)$$

For $n = 1$,

$$\begin{aligned} |y_1(x) - y_0(x)| &\leq M|x - x_0|, \\ &\leq Mh < k. \end{aligned}$$

For $n - 1$: Let

$$|y_{n-1}(x) - y_{n-2}(x)| \leq \frac{MA^{n-2}}{(n-1)!} |x - x_0|^{n-1}, \quad (12)$$

and from equation (10), we have that

$$\begin{aligned} |y_n(x) - y_{n-1}(x)| &= \left| \int_{x_0}^x (f(x, y_{n-2}(x)) - f(x, y_{n-1}(x)))dx \right|, \\ |y_n(x) - y_{n-1}(x)| &\leq \int_{x_0}^x |f(x, y_{n-2}(x)) - f(x, y_{n-1}(x))| |dx|. \end{aligned} \quad (13)$$

Lipschitz condition in equation (8) gives:

$$|f(x, y_{n-2}(x)) - f(x, y_{n-1}(x))| < A|y_{n-1}(x) - y_{n-2}(x)|. \quad (14)$$

From equation (13) and (14), we have;

$$\begin{aligned} |y_n(x) - y_{n-1}(x)| &\leq \int_{x_0}^x A|y_{n-1}(x) - y_{n-2}(x)| |dx|, \\ |y_n(x) - y_{n-1}(x)| &\leq A \int_{x_0}^x |y_{n-1}(x) - y_{n-2}(x)| |dx|. \end{aligned} \quad (15)$$

And from equation (12)

$$\begin{aligned} |y_n(x) - y_{n-1}(x)| &\leq A \int_{x_0}^x \frac{MA^{n-2}}{(n-1)!} |x - x_0|^{n-1} |dx|, \end{aligned}$$

$$|y_n(x) - y_{n-1}(x)| \leq A \frac{MA^{n-2}}{(n-1)!} \int_{x_0}^x |x - x_0|^{n-1} |dx|,$$

$$|y_n(x) - y_{n-1}(x)| \leq \frac{MA^{n-1}}{(n-1)!} \int_{x_0}^x (x - x_0)^{n-1} dx,$$

$$|y_n(x) - y_{n-1}(x)| \leq \frac{MA^{n-1}}{(n-1)!} \frac{(x - x_0)^n}{n}.$$

Thus by induction,

$$|y_n(x) - y_{n-1}(x)| \leq \frac{MA^{n-1}}{n!} |x - x_0|^n \quad \forall n \in N. \quad (16)$$

Third step: Now we will show that the sequence $y_n(x)$ converges uniformly to a limit $x_0 - h \leq x \leq x_0 + h$ for interval

$$|x - x_0| \leq h. \quad (17)$$

Substitute equation (17) into (16),

$$|y_n(x) - y_{n-1}(x)| \leq \frac{MA^{n-1}}{n!} h^n. \quad (18)$$

By infinite series and the fact that the difference is bounded, equation 18 becomes:

$$y_0(x) + (y_1(x) - y_0(x)) + (y_2(x) - y_1(x)) + \dots + (y_n(x) - y_{n-1}(x)) + \dots$$

$$\leq y_0 + Mh + \frac{1}{2!} Mkh^2 + \dots + \frac{1}{n!} Mk^{n-1} h^n + \dots,$$

$$y_0(x) + (y_1(x) - y_0(x)) + (y_2(x) - y_1(x)) + \dots + (y_n(x) - y_{n-1}(x)) + \dots \leq y_0 + \frac{M}{k} (e^{kh} - 1). \quad (19)$$

Consequently, the series in equation (19) is absolutely convergent for all h . Hence by Weierstrass M-test, the series converges uniformly for $x_0 - h \leq x \leq x_0 + h$. Since the terms of the series are continuous in x , then the sum $\lim_{n \rightarrow \infty} y_n(x) = y(x)$ is also continuous.

Fourth step: We will now show that $y = y(x)$ satisfies the differential equation $y' = f(x, y)$. Since $y_n(x)$ converges uniformly to $y(x)$ in $[x_0 - h, x_0 + h]$ by Lipschitz condition (8), we then have that

$$|f(x, y(x)) - f(x, y_n(x))| \leq A|y(x) - y_n(x)|$$

It follows that $f(x, y_n(x))$ tends uniformly to $f(x, y(x))$. If we let $n \rightarrow \infty$ in equation (10), we will then have:

$$\lim_{n \rightarrow \infty} y_n(x) = y_0 + \lim_{n \rightarrow \infty} \int_{x_0}^x f(x, y_{n-1}(x)) dx, \quad (20)$$

$$y(x) = y_0 + \int_{x_0}^x \lim_{n \rightarrow \infty} f(x, y_{n-1}(x)) dx. \quad (21)$$

Since $f(x, y_n(x))$ is a sequence which consists of a continuous function which converges uniformly to $f(x, y(x))$ on the same interval, hence

$$y(x) = y_0 + \int_{x_0}^x f(x, y(x)) dx. \quad (22)$$

The integrand $f(x, y(x))$ is a continuous function of x . Therefore the integral has the derivative $f(x, y)$ and $y(x_0) = y_0$.

Finally, we will show that the solution $y = y(x)$ is unique and $y(x_0) = y_0$. Lets assume $y = Y(x)$ so that,

$$|Y(x) - y(x)| \leq c, \quad (23)$$

where $x_0 - h \leq x \leq x_0 + h$ and $c = 2k$. From equation (22), we have;

$$|Y(x) - y(x)| = \left| \int_{x_0}^x f(x, Y(x)) - f(x, y(x)) dx \right|,$$

$$|Y(x) - y(x)| = A \int_{x_0}^x |Y(x) - y(x)| dx, \quad (24)$$

$$|Y(x) - y(x)| \leq Ac|x - x_0|. \quad (25)$$

Now substituting equation (22) into (24), to have;

$$|Y(x) - y(x)| \leq A^2 c \int_{x_0}^x |x - x_0| dx,$$

$$|Y(x) - y(x)| \leq \frac{A^2 c |x - x_0|^2}{2!}. \quad (26)$$

Again substituting equation (26) into (24) gives;

$$|Y(x) - y(x)| \leq \frac{A^3 c}{3!} |x - x_0|^3,$$

and recursively becomes

$$|Y(x) - y(x)| \leq \frac{c(Ah)^n}{n!}, \text{ where } |x - x_0| = h.$$

But the series $\sum_{k=0}^n \frac{c(Ah)^k}{k!}$ converges so that $\lim_{n \rightarrow \infty} \frac{c(Ah)^n}{n!} = 0$. Thus $|Y(x) - y(x)| \rightarrow 0$ and we conclude that $Y(x) = y(x)$. Which shows that $y = y(x)$ is unique.

3. ADOMIAN DECOMPOSITION METHOD (ADM)

The Adomian decomposition method (ADM) is an approximation method used for solving non-linear differential equation, [Ado88]. This method involves

the decomposition of a non-linear equations into solutions in a series of functions. These series of functions are obtained in a recursive procedure from a polynomial called Adomian polynomials which are initiated from the expansion of an analytic function into a power series function [Abd08]. The method will be analysed and we will prove its convergence.

3.1. Analysis of the Adomian Decomposition Method

In this section, we present the derivation of the Adomian decomposition method. Consider the equation

$$y' + f(x, y) = g(x), \quad y(0) = A, \quad (27)$$

where $f(x, y)$ is real, non-linear and continuous, $g(x) \in C[0,1]$ are linear functions, and A is constant. We define the differential operator L as

$$L = \frac{d}{dx}$$

and

$$L_y = \frac{d}{dx}(y) = y'.$$

Since

$$L_y = y'. \quad (28)$$

Hence equation (27) becomes

$$L_y + f(x, y) = g(x), \quad (29)$$

which can also be written as

$$L_y = g(x) - f(x, y). \quad (30)$$

And the inverse operator which is considered as a linear integral operator (n-fold integral) is given as

$$L^{-1}(\cdot) = \int_0^x (\cdot) dx. \quad (31)$$

When we find the inverse of L_y in equation (30), it gives:

$$L^{-1}(L_y) = L^{-1}(g(x)) - L^{-1}(f(x, y)). \quad (32)$$

Substituting L_y from (28) into (32), to obtain

$$L^{-1}(y') = L^{-1}(g(x)) - L^{-1}(f(x, y)). \quad (33)$$

Evaluating the left hand side of equation (33), we obtain:

$$L^{-1}(y') = \int_0^x (y') dx.$$

Computing the integral and using the initial value condition given to obtain

$$L^{-1}(y') = \int_0^x (y') dx = y(x) + A, \quad (34)$$

where A is a constant of integration. Substituting equation (34) into (33),

$$y(x) + A = L^{-1}(g(x)) - L^{-1}(f(x, y)),$$

hence,

$$y(x) = A - L^{-1}(g(x)) + L^{-1}(f(x, y)). \quad (35)$$

The Adomian decomposition method introduces the solution $y(x)$ in an infinite series form;

$$y(x) = \sum_{n=0}^{\infty} y_n(x), \quad (36)$$

The components of $y_n(x)$ are determined recursively. The Adomian decomposition method also defines the non-linear function by an infinite polynomial series of the form:

$$f(x, y) = \sum_{n=0}^{\infty} A_n, \quad (37)$$

The A_n 's are called the Adomian polynomials which can be constructed using the general formula for a non-linear function $F(u)$,

$$A_n = \frac{1}{n!} \left[\frac{d^n}{d\lambda^n} \left[F \left(\sum_{i=0}^n \lambda^i u^i \right) \right] \right]_{\lambda=0}, \quad n = 0, 1, 2, \dots,$$

where λ is a parameter introduced for convenience. Constructing the first four iterations for A_n ,

$$\begin{aligned} A_0 &= F(u_0), \\ A_1 &= \left[\frac{d}{d\lambda} F(u_0 + u_1 \lambda) \right]_{\lambda=0} = u_1 F'(u_0), \\ A_2 &= \frac{1}{2!} \frac{d}{d\lambda} [(u_1 + 2u_2 \lambda) F'(u_0 + u_1 \lambda)]_{\lambda=0} \\ &= u^2 F'(u_0) + \frac{1}{2!} u_1^2 F''(u_0). \end{aligned}$$

Similarly,

$$\begin{aligned} A_3 &= u_3 F'(u_0) + u_1 u_2 F''(u_0) + \frac{1}{3!} u_1^3 F'''(u_0), \\ A_4 &= u_4 F'(u_0) + (u_1 u_3 + \frac{1}{2!} u_2^2) F''(u_0) \\ &\quad + \frac{1}{2!} u_1^2 u_2 F'''(u_0) + \frac{1}{4!} F^4(u_0). \end{aligned}$$

Substituting equations (36) and (37) in (35) gives:

$$\sum_{n=0}^{\infty} y_n(x) = A - L^{-1}(g(x)) - L^{-1}(\sum_{n=0}^{\infty} A_n) \quad (38)$$

Now, to determine the component of $y_n(x)$, we use the Adomian decomposition method which makes use of the recursive relation:

$$\begin{aligned} y_0(x) &= A + L^{-1}(g(x)), \\ y_{k+1}(x) &= -L^{-1}(A_k), \quad k \geq 0. \end{aligned} \quad (39)$$

which gives:

$$\begin{aligned} y_0(x) &= A + L^{-1}(g(x)), \\ y_1(x) &= -L^{-1}(A_0), \\ y_2(x) &= -L^{-1}(A_1), \\ y_3(x) &= -L^{-1}(A_2), \\ y_4(x) &= -L^{-1}(A_3), \\ &\vdots \\ y_{k+1}(x) &= -L^{-1}(A_k). \end{aligned}$$

From the recursive relation above, the components of $y_n(x)$ can be determined and hence the series solution obtained. For numerical purposes, the n-term approximation is given by,

$$\varphi_n = \sum_{j=0}^{n-1} y_j,$$

where the series obtained will converge to that of the exact solution given, then

$$y(x) = \lim_{n \rightarrow \infty} \sum_{n=0}^{n-1} y_n(x). \quad (40)$$

4. CONVERGENCE OF THE ADM

In this section, the order and rate of convergence will be defined and also, the convergence of the ADM will be proved.

Order and rate of convergence

For every $n \in \mathbb{N} \setminus \{0\}$, we define the rate of convergence as ([MS12]):

$$\alpha_n = \begin{cases} \frac{\|y_{n+1}\|}{\|y_n\|}, & \|y_n\| \neq 0, \\ 0, & \|y_n\| = 0. \end{cases} \quad (41)$$

Theorem 4.1

Let N be an operator from a Hilbert space H into H and y be the exact solution of (42). Then the sum, $\sum_{n=0}^{\infty} y_n$, which is obtained from (38) converges to y when $\exists 0 \leq \alpha < 1$, $\|y_{n+1}\| \leq \alpha \|y_n\|$, $\forall n \in \mathbb{N} \setminus \{0\}$ [MS12].

Corollary 4.2 The expression in theorem (4.1) i.e. $\sum_{n=0}^{\infty} y_n$ converges to an exact solution y , when $0 \leq \alpha_n < 1$, $n = 1, 2, 3, \dots$ [MS12].

Proof. A detailed proof of the convergence of ADM using the entire series property as shown by [CSS92]. Consider a non-linear functional equation

$$y = N(y) + f, \quad (42)$$

since $y(x)$ can be written as a series in an infinite form

$$y = \sum_{n=0}^{\infty} y_n.$$

This expression can be rewritten by introducing λ , the λ is used to test for the convergence so that we have:

$$y\lambda = \sum_{n=0}^{\infty} y_n \lambda^n. \quad (43)$$

Let's suppose that the convergence radius γ of (43) is greater than 1 i.e. $\gamma > 1$, we then conclude that (43) converges when $|\lambda| \leq \gamma$. Assume $N(y)$ can be written in an entire series form:

$$N(y) = \sum_{i=0}^{\infty} g_i y^i, \quad (44)$$

with radius $\rho > 1$, so that the series converges for $|y| < \rho$. If we let $\rho = +\infty$, it is possible that the non-linear function $N(y)$ converges for any y with $|y| < +\infty$. Suppose we let:

$$r = p(y) = a_0 + a_1 y + a_2 y^2 + \dots + a_n y^n + \dots, \quad (45)$$

be a series which converges for $|y| < +\infty$, and we also let

$$y = q(x) = b_0 + b_1 x + b_2 x^2 + \dots + b_n x^n + \dots, \quad (46)$$

be another series which converges for $|x| \leq k$, ($k > 1$). Substituting equation (46) into (45) we have:

$$\begin{aligned} r &= a_0 + a_1(b_0 + b_1 x + b_2 x^2 + \dots) + a_2(b_0 + b_1 x \\ &\quad + b_2 x^2 + \dots)^2 + \dots \\ &= a_0 + a_1 b_0 + a_1 b_1 x + a_1 b_2 x^2 + a_2 b_0^2 \\ &\quad + 2a_2 b_0 b_1(x) + a_2 b_1 x^2 \\ &\quad + a_2 2b_0 b_2 x^2 + \dots, \end{aligned}$$

so that we have:

$$\begin{cases} c_0 = a_0 + a_1 b_0 + a_2 b_0^2 + \dots, \\ c_1 = a_1 b_1 + 2a_2 b_1 b_0 + \dots, \\ c_2 = a_1 b_2 + a_2(b_1^2 + 2b_0 b_2) + \dots. \end{cases} \quad (47)$$

which can be written in series $\sum_{i=0}^{\infty} c_n x^n$, the radius

of convergence of this series is strictly greater than 1 i.e. the series converges for $|\sigma| > 1$. Putting these results into $N(y) = \sum_{i=0}^{\infty} g_i y^i$, since

$$y = \sum_{n=0}^{\infty} y_n \lambda^n \text{ and } N(y) = \sum_{i=0}^{\infty} g_i y^i,$$

which can be written as

$$N(y) = \sum_{i=0}^{\infty} g_i (\sum_{n=0}^{\infty} y_n \lambda^n)^i$$

so that we obtain a series

$$N(y) = \sum_{i=0}^{\infty} A_i \lambda^i \quad (48)$$

which converges for $|\lambda| \leq \rho, (\rho > 1)$, where the A_i 's depend on the u_0, u_1, \dots, u_n which are determined from the relationship in (47). Now substituting (43) and (44) into (42), and assume that $\lambda = 1$ so that we obtain:

$$\sum_{i=0}^{\infty} y_i - \sum_{i=0}^{\infty} A_i = f. \quad (49)$$

This expression is satisfied by the Adomian's equation [CSS92]

$$\begin{cases} y_0 = f, \\ y_1 = A_0, \\ \vdots \\ y_n = A_{n-1}. \\ \vdots \end{cases} \quad (50)$$

The proof is concluded by checking that the non-linear function has a series solution which remains stable under perturbation i.e. $\sum_{n=0}^{\infty} (1 + \epsilon)^n |y_n| < \infty$, for $\epsilon > 0$ very small. And also that the non-linear operator $N(y)$ can be developed in series according to y .

5. NUMERICAL SIMULATION AND DISCUSSION

In this section, numerical experiments are performed on some illustrative examples of some non-linear ordinary differential equations to check the difference between the methods described. The performance of the methods are checked in terms of accuracy, efficiency and convergence.

5.1. Example 1

We will now consider the following non-linear differential equation;

$$y' = y^2, \quad y(0) = 1. \quad (51)$$

With exact solution of $y(x) = \frac{1}{1-x}$.

PIM Solution

Using the same approach as that of example 1 above, the first three solutions of the iteration are given as;

$$\begin{aligned} y_0 &= 1, \\ y_1 &= 1 + x, \\ y_2 &= 1 + x + x^2 + \frac{1}{3}x^3, \\ y_3 &= 1 + x + x^2 + x^3 + \frac{2}{3}x^4 + \frac{1}{3}x^5 + \frac{1}{9}x^6 + \frac{1}{63}x^7 \end{aligned} \quad (52)$$

ADM Solution

Using the same approach as that of example 1 above, the first three solutions are;

$$\begin{aligned} y_0 &= 1, \\ y_1 &= x, \\ y_2 &= x^2, \\ y_3 &= x^3. \end{aligned} \quad (53)$$

Hence, the approximate solution for $y(x)$ obtained using $n = 0, 1, 2, \dots, 7$ is given by;

$$\phi_8(x) = 1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7. \quad (54)$$

Table 1: Approximation of $y(x)$ for the PIM at selected values of x and orders

x	2nd order	4th order	6th order	8th order	Exact
.0	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000
.1	1.11033333	1.11110938	1.11111111	1.11111111	1.11111111
.2	1.24266667	1.24992670	1.24999965	1.25000000	1.25000000
.3	1.39900000	1.42781460	1.42856219	1.42857136	1.42857143
.4	1.58133333	1.66219250	1.66655381	1.66666502	1.66666667
.5	1.79166667	1.98007633	1.99906325	1.99997461	2.00000000
.6	2.03200000	2.42378080	2.49363107	2.49969413	2.50000000
.7	2.30433333	3.06008988	3.29319138	3.32991902	3.33333333
.8	2.61066667	3.99493989	4.73537940	4.95796008	5.00000000
.9	2.95300000	5.39676995	7.72314683	9.22167456	10.00000000
Run Time(Secs)	0.031	0.131	1.147	2.262	

Table 2: Approximation of $y(x)$ for the ADM at selected values of x and orders

x	2nd order	4th order	6th order	8th order	Exact
.0	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000
.1	1.11000000	1.11110000	1.11111100	1.11111111	1.11111111
.2	1.24000000	1.24960000	1.24998400	1.25000000	1.25000000
.3	1.39000000	1.42510000	1.42825900	1.42854331	1.42857143
.4	1.56000000	1.64960000	1.66393600	1.66622976	1.66666667
.5	1.75000000	1.93750000	1.98437500	1.99609375	2.00000000
.6	1.96000000	2.30560000	2.43001600	2.47480576	2.50000000
.7	2.19000000	2.77310000	3.05881900	3.19882131	3.33333333
.8	2.44000000	3.36160000	3.95142400	4.32891136	5.00000000
.9	2.71000000	4.09510000	5.21703100	6.12579511	10.00000000
Run Time (Secs)	0.017	0.032	0.32	0.044	

Table 3: Error Estimation of $y(x)$ using PIM for selected values of x and orders

x	2nd order	4th order	6th order	8th order
.0	0.00000000	0.00000000	0.00000000	0.00000000
.1	0.00077778	0.00000173	0.00000000	0.00000000
.2	0.00733333	0.00007330	0.00000035	0.00000000
.3	0.02957143	0.00075682	0.00000924	0.00000007
.4	0.08533333	0.00447416	0.00011286	0.00000165
.5	0.20833333	0.01992367	0.00093675	0.00002539
.6	0.46800000	0.07621920	0.00636893	0.00030587
.7	1.02900000	0.27324345	0.04014196	0.00341431
.8	2.38933333	1.00506011	0.26462060	0.04203992
.9	7.04700000	4.60323005	2.27685317	0.77832544

Table 4: Error Estimation of $y(x)$ using ADM for the selected values of x and orders

x	2nd order	4th order	6th order	8th order
.0	0.00000000	0.00000000	0.00000000	0.00000000
.1	0.00111111	0.00001111	0.00000011	0.00000000
.2	0.01000000	0.00040000	0.00001600	0.00000064
.3	0.03857143	0.00347143	0.00031243	0.00002812
.4	0.10666667	0.01706667	0.00273067	0.00043691
.5	0.25000000	0.06250000	0.01562500	0.00390625
.6	0.54000000	0.19440000	0.06998400	0.02519424
.7	1.14333333	0.56023333	0.27451433	0.13451202
.8	2.56000000	1.63840000	1.04857600	0.67108864
.9	7.29000000	5.90490000	4.78296900	3.87420489

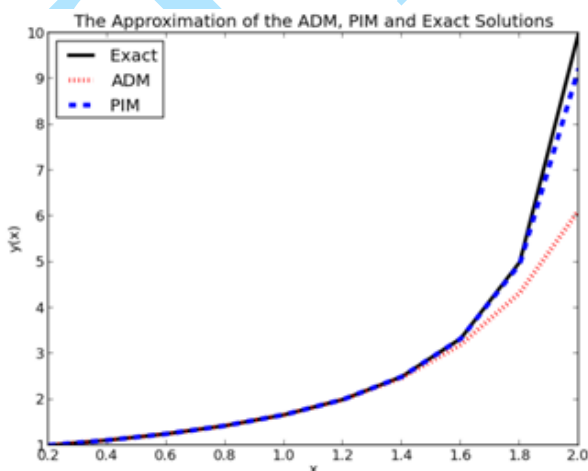


Figure 1: Approximation of $y(x)$ for the ADM, PIM and the Exact Solution at the selected values of x considering the 8th order

Tables 1 and 2 shows the comparison between results of the PIM, ADM and the exact solutions computed for different orders and some selected values of x and their run times. The absolute error between the PIM and the exact solutions; the ADM and the exact solutions are shown in tables 3 and 4 respectively. Figure 1 shows the comparison between the PIM, ADM and the exact solution at the 8th order. From table 3, we can see that the error rate are minimal for small values of x , accuracy and convergence of the PIM results generally increases when the number of iterations is increased. We can see from table 4 that the error approximation is large for the 2nd and 4th order, but on getting to the 8th order, the error is minimal for small values of x . We can then say that the ADM is less accurate and

converges slower to the exact solutions. When comparing between the two methods using figure 1, we can see that the PIM is more accurate and converges to the exact solutions faster than the ADM, but the ADM is more efficient than the PIM since the run time estimate for the PIM is much more greater than that of the ADM.

5.2. Example 2

Let's consider the following non-linear differential equation.

$$y' + y^2 = 1, \quad y(0) = 0. \quad (55)$$

With exact solution of $y(x) = \frac{1-e^{-2x}}{1+e^{-2x}}$.

PIM Solution

The first four solution are given by;

$$\begin{aligned} y_0 &= 0, \\ y_1 &= x, \\ y_2 &= x - \frac{1}{3}x^3, \end{aligned}$$

$$\begin{aligned} y_3 &= x - \frac{1}{3}x^3 - \frac{1}{63}x^7 + \frac{2}{15}x^{15}, \\ y_4 &= x - \frac{1}{3}x^3 + \frac{2}{15}x^5 - \frac{17}{315}x^7 + \frac{38}{2835}x^9 - \\ &\quad - \frac{134}{51975}x^{11} + \frac{4}{12285}x^{13} - \frac{1}{59535}x^{15}. \end{aligned} \quad (56)$$

ADM Solution

The first four solutions are;

$$\begin{aligned} y_0 &= x, \\ y_1 &= -\frac{1}{3}x^3, \\ y_2 &= \frac{2}{15}x^5, \\ y_3 &= -\frac{17}{315}x^7, \\ y_4 &= \frac{62}{2835}x^9. \end{aligned} \quad (57)$$

Hence, the approximate solution for $y(x)$ obtained using $n = 1, 2, \dots, 7$ is given by;

$$\begin{aligned} \phi_8(x) &= x - \frac{1}{3}x^3 + \frac{2}{15}x^5 - \frac{17}{315}x^7 + \frac{62}{2835}x^9 - \\ &\quad - \frac{1382}{155925}x^{11} + \frac{21844}{6081075}x^{13} - \frac{929569}{638512875}x^{15}. \end{aligned} \quad (58)$$

Table 5: Approximation of $y(x)$ for the PIM at selected values of x and orders

x	2nd order	4th order	6th order	8th order	Exact
.2	0.19733333	0.19737532	0.19737532	0.19737532	0.19737532
.4	0.37866667	0.37994699	0.37994896	0.37994896	0.37994896
.6	0.52800000	0.53698338	0.53704935	0.53704957	0.53704957
.8	0.62933333	0.66330092	0.66402980	0.66403673	0.66403677
.0	0.66666667	0.75716627	0.76150059	0.76159305	0.76159416
.2	0.62400000	0.81562802	0.83293984	0.83363872	0.83365461
.4	0.48533333	0.82929787	0.88164769	0.88521413	0.88535165
.6	0.23466667	0.77889309	0.90719341	0.92084260	0.92166855
.8	-0.14400000	0.63789308	0.90121266	0.94303954	0.94680601
.0	-0.66666667	0.39169207	0.84363640	0.95067963	0.96402758
Run Time (Secs)	0.034	0.096	1.052	2.593	

Table 6: Approximation of $y(x)$ for the ADM at selected values of x and orders

x	2nd order	4th order	6th order	8th order	Exact
.2	0.19737600	0.19737532	0.19737532	0.19737532	0.19737532
.4	0.38003200	0.37994931	0.37994896	0.37994896	0.37994896
.6	0.53836800	0.53707763	0.53705016	0.53704958	0.53704957
.8	0.67302400	0.66464131	0.66407744	0.66403951	0.66403677
.0	0.80000000	0.76790123	0.76263013	0.76176432	0.76159416
.2	0.95577600	0.87523975	0.84781849	0.83847884	0.83365461
.4	1.20243200	1.08538025	1.01157026	0.96499619	0.88535165
.6	1.63276800	1.68692852	1.74544229	1.80843315	0.92166855
.8	2.37542400	3.40939561	5.19301064	8.26849728	0.94680601
.0	3.60000000	7.88924162	19.164044813	48.79534367	0.96402758
Run Time (Secs)	0.014	0.016	0.023	0.041	

Table 7: Error Estimation of $y(x)$ using PIM for selected values of x and orders

x	2nd order	4th order	6th order	8th order
.2	0.00004199	0.00000000	0.00000000	0.00000000
.4	0.00128229	0.00000197	0.00000000	0.00000000
.6	0.00904957	0.00006619	0.00000022	0.00000000
.8	0.03470344	0.00073585	0.00000697	0.00000004
.0	0.09492749	0.00442789	0.00009357	0.00000111
.2	0.20965461	0.01802659	0.00071477	0.00000159
.4	0.40001832	0.05605378	0.00370396	0.00013752
.6	0.68700188	0.14277546	0.01447514	0.00082595
.8	1.09080601	0.30891293	0.04559335	0.00376647
.0	1.63069425	0.572335519	0.12039118	0.01334795

Table 8: Error Estimation of $y(x)$ using ADM for selected values of x and orders

x	2nd order	4th order	6th order	8th order
.2	0.00000068	0.00000000	0.00000000	0.00000000
.4	0.00008304	0.00000035	0.00000000	0.00000000
.6	0.00131843	0.00002806	0.00000060	0.00000001
.8	0.00899723	0.00060454	0.00004067	0.00000274
.0	0.03840584	0.00630708	0.00103597	0.00017016
.2	0.12212139	0.04158514	0.01416389	0.00482423
.4	0.31708035	0.20002860	0.12621861	0.07964454
.6	0.71109945	0.76525997	0.82377373	0.88676459
.8	1.42861799	2.46258960	4.24620463	7.32169126
.0	2.63597242	6.92521404	18.20002055	47.83131609

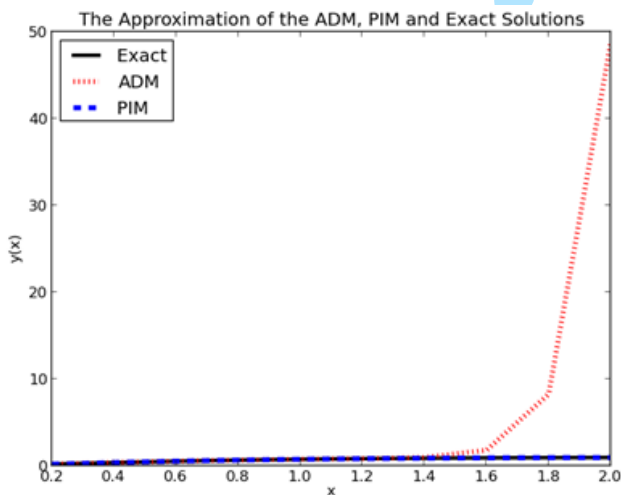


Figure 2: Approximation of $y(x)$ for the ADM, PIM and the Exact Solution at the selected values of x considering the 8th order

Tables 5 and 6 shows the comparison between results of the PIM, ADM and the exact solutions computed for different orders and some selected values of x and their run times. The absolute error between the PIM and the exact solutions; the ADM and the exact solutions are shown in tables 7 and 8 respectively. Figure 2 shows the comparison between the PIM, ADM and the exact solutions at the 8th order. From table 7, we can see that the error rate are minimal for small values of x , accuracy and convergence of the PIM results generally increases

when the number of iterations is increased. We can see from table 8 that the error approximation is large for the 2nd and 4th order, but on getting to the 8th order, the error is minimal for small values of x . From figure 2, we can then say that the ADM is less accurate and converges slower to the exact solutions, infact, it converges when $0 < x < 1$ but when $x > 1$ it diverges. When comparing between the two methods, we can infer that the PIM is more accurate and converges to the exact solutions faster than the ADM for this problem but the ADM is more efficient than the PIM since the run time estimate for the PIM is much more greater than that of the ADM.

5.3. Example 3

Consider the third order non-linear differential equation;

$$y''' + yy'' - (y')^2 - 3y' = 0, \quad y(0) = 0, y'(0) = 1, y''(0) = -2. \quad (59)$$

With exact solution of $y(x) = \frac{1}{2}(1 - e^{-2x})$.

PIM Solution

Solving this third order equation, the PIM iterates are obtained by;

$$y_0(x) = \int_0^x \int_0^x \int_0^x (y''') dx dx dx,$$

$$y_{n+1}(x) = y_0 + \int_0^x \int_0^x \int_0^x (-yy'' + (y')^2 + 3y') dx dx dx.$$

Thus, the first four solutions are given by;

$$\begin{aligned} y_0 &= x - x^2, \\ y_1 &= x - x^2 + \frac{1}{2}x^3 - \frac{1}{30}x^5 - \frac{1}{6}x^6 + \frac{1}{945}x^9, \\ y_2 &= x - x^2 + \frac{46513}{69120}x^3 - \frac{11813}{34560}x^4 + \frac{1231}{10800}x^5 - \\ &\frac{2539}{86400}x^6 + \frac{1}{945}x^9 - \frac{1}{8100}x^{10}. \end{aligned} \quad (60)$$

ADM Solution

Solving this third order non-linear differential equation using ADM, the linear operator L is defined as

$$L = \frac{d^3}{dx^3}, \quad (61)$$

$$L_y = -yy'' + (y')^2 + 3y, \quad (62)$$

with inverse operator

$$L^{-1}() = \int_0^x \int_0^x \int_0^x () dx dx dx. \quad (63)$$

Applying L^{-1} on both sides of (62), we have;

$$y = y_0 - L^{-1}(-yy'' + (y')^2 + 3y'), \quad (64)$$

and we obtain the following recurrence relation:

$$\begin{aligned} y_0(x) &= x - x^2, \\ y_{n+1}(x) &= L^{-1}(A_n), n \geq 0. \end{aligned} \quad (65)$$

Thus, the first four solutions are given by;

$$\begin{aligned} y_0 &= x - x^2, \\ y_1 &= -\frac{2}{3}x^3 + \frac{1}{3}x^4 - \frac{1}{30}x^5, \\ y_2 &= \frac{1}{480} - \frac{1}{48}x + \frac{1}{12}x^2 + \frac{1}{60}x^3 - \frac{17}{576}x^4 \\ &\quad + \frac{1}{180}x^5 - \frac{1}{45}x^6 - \frac{1}{252}x^7 \\ &\quad + \frac{1}{5040}x^8, \\ y_3 &= -\frac{1}{69120}x^3 - \frac{7}{34560}x^4 - \frac{7}{2700}x^5 + \frac{119}{28800}x^6 - \\ &\frac{3}{320}x^7 + \frac{67}{20160}x^8 - \frac{1}{3360}x^9. \end{aligned} \quad (66)$$

Hence, the approximate solution for $y(x)$ obtained using $n = 1, 2, \dots, 4$ is given by;

$$\begin{aligned} \phi_4(x) &= \frac{1}{480} + \frac{47}{48}x - \frac{11}{12}x^2 - \frac{44929}{69120}x^3 + \\ &\frac{104933}{345600}x^4 + \frac{15433}{4838400}x^8 - \frac{101}{129729600}x^{13} + \\ &\frac{317}{908107200}x^{14}. \end{aligned} \quad (67)$$

Table 9: Approximation of $y(x)$ for the PIM at selected values of x and orders

x	2nd order	4th order	6th order	Exact
.2	0.16487017	0.16483999	0.16483998	0.16483998
.4	0.27537131	0.27533578	0.27533552	0.27533552
.6	0.34865541	0.34940469	0.34940289	0.34940289
.8	0.39489498	0.39905998	0.39905174	0.39905174
.0	0.41872599	0.43235977	0.43233237	0.43233236
.2	0.41964656	0.45470844	0.45464105	0.45464102
.4	0.39158272	0.46970792	0.46959507	0.46959497
.6	0.32194159	0.47967461	0.47961934	0.47961890
.8	0.19058250	0.48586944	0.48634005	0.48633814
.0	-0.03073806	0.48842367	0.49084946	0.49084218
RunTime (Secs)	0.014	0.141	3.089	

Table 10: Approximation of $y(x)$ for the ADM at selected values of x and orders

x	2nd order	4th order	6th order	Exact
.2	0.16367776	0.16367889	0.16367889	0.16483998
.4	0.26860792	0.26863748	0.26863749	0.27533552
.6	0.32983248	0.33008788	0.33008808	0.34940289
.8	0.35722020	0.35858265	0.35858465	0.39905174
.0	0.35451389	0.35984133	0.35985512	0.43233236
.2	0.31927838	0.33589375	0.33596788	0.45464102
.4	0.24199342	0.28562702	0.28596091	0.46959497
.6	0.10463972	0.20475946	0.20606866	0.47961890
.8	-0.12085328	0.08508268	0.08964179	0.48633814
.0	-0.47390873	-0.08741669	-0.07314903	0.49084218
Run Time (Secs)	0.009	0.016	0.094	

Table 11: Error Estimation of $y(x)$ using PIM for selected values of x and orders

x	2nd order	4th order	6th order
.2	0.00003019	0.00000001	0.00000000
.4	0.00003579	0.00000026	0.00000000
.6	0.00074748	0.00000180	0.00000000
.8	0.00415676	0.00000824	0.00000000
.0	0.01360637	0.00002742	0.00000001
.2	0.03499446	0.00006742	0.00000003
.4	0.07801225	0.00011295	0.00000010
.6	0.15767731	0.00005571	0.00000044
.8	0.29575564	0.00046870	0.00000191
.0	0.502158024	0.00241851	0.00000728

Table 12: Error Estimation of $y(x)$ using ADM for selected values of x and orders

x	2nd order	4th order	6th order
.2	0.00116221	0.00116108	0.00116108
.4	0.00672760	0.00669804	0.00669802
.6	0.01957041	0.01931501	0.01931482
.8	0.04183154	0.04046909	0.04046709
.0	0.07781847	0.07249103	0.07247724
.2	0.13536265	0.11874727	0.11867315
.4	0.22760155	0.18396795	0.18363405
.6	0.37497918	0.27485944	0.27355024
.8	0.60719141	0.40125546	0.39669635
.0	0.96475091	0.57825887	0.56399121

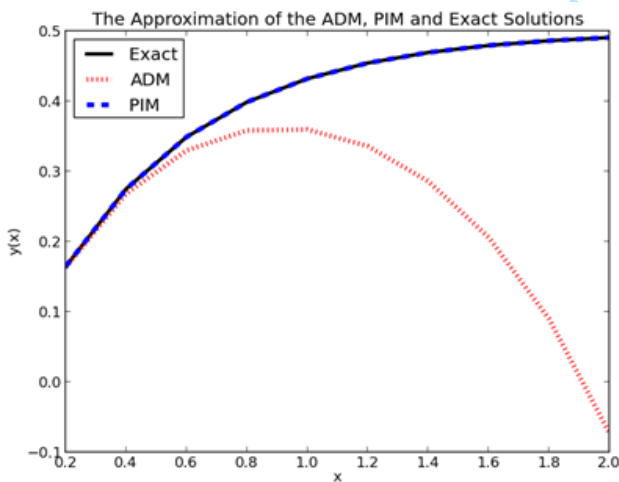


Figure 3: Approximation of $y(x)$ for the ADM, PIM and the Exact Solution at the selected values of x considering the 6th order

Tables 9 and 10 shows the comparison between results of the PIM, ADM and the exact solutions computed for different orders and some selected values of x and their run times. The absolute error between the PIM and the exact solutions; the ADM and the exact solutions are shown in tables 11 and 12 respectively. Figure 3 shows the comparison between the PIM, ADM and the exact solutions at the 8th order. From table 11, we can see that the error rate are minimal for small values of x ,

accuracy and convergence of the PIM results generally increases when the number of iterations is increased. We can see from table 12 that the error approximation is large for the 2nd and 4th order, but on getting to the 6th order, the error is minimal for small values of x . From figure 3, we can then say that the ADM is less accurate and converges slower to the exact solutions, infact, it converges when $0 < x < 1$ but when $x > 1$ it diverges. When comparing between the two methods, we can infer that the PIM is more accurate and converges to the exact solutions faster than the ADM for this problem but the ADM is more efficient than the PIM since the run time estimate for the PIM is much more greater than that of the ADM.

ACKNOWLEDGMENT

The authors are grateful to the reviewers and the handling editor for their helpful comments and suggestions that had greatly improved this work.

REFERENCES

- [Abd08] **A. H. M. Abdelrazec** – *Adomian decomposition method: convergence analysis and numerical approximations*, M.Sc. Thesis, McMaster University, Hamilton,

- Ontario, 2008. [G+12] **B. Gbadamosi, O. Adebimpe, E. I. Akinola, I. A. Olopade** – *Solving Riccati Equation Using Adomian Decomposition Method*, International Journal of Pure and Applied Mathematics, Vol. 78, No. 3, pp. 409-417, 2012.
- [Ado88] **G. Adomian** – *A review of the decomposition in applied mathematics*, Mathematical analysis and applications, Vol. 135, pp. 501-544, 1988.
- [Ado92] **G. Adomian** – *A review of the decomposition method and some recent results for nonlinear equation*, Mathematical Computational Model., Vol. 13(7), 1992. [Lin94] **E. Lindelof** – *Sur l'application de la méthode des approximations successives aux équations différentielles ordinaires du premier ordre*. Comptes rendus hebdomadaires des séances de l'Académie des sciences. 116: 454-457, 1894.
- [AR96] **G. Adomian, R. Rach** – *Modified Adomian Polynomials*, Mathematical Computational Model., Vol. 24, No. 11, pp. 39-46, 1996. [MS12] **S. S. Motsa, S. Shateyi** – *New Analytic Solution to the Lane-Emden Equation of Index 2*, Mathematical problems in Engineering, pp. 1-20, 2012.
- [CL55] **Coddington E. A., Levinson N.** – *Theory of Ordinary Differential Equations*. New York: McGraw-Hill, 1955. [NSS04] **R. K. Nagle, E. B. Saff, A. D. Snider** – *Fundamentals of differential equations and boundary value problems*, 4th Edition, Pearson Addison Wesley, 2004.
- [CSS92] **Y. Cherruault, G. Saccomandi, B. Some** – *New results for convergence of Adomian's method applied to integral equations*, Mathematical computational modelling, Vol. 16, No. 2, pp. 85-93, 1992.