Abstract Mobius-Division Categories are Reduced Standard Division Categories of Combinatorial Inverse Monoids

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ABSTRACT. A main result of [Sch05,Theorem 3.3, Section 3.] is detailed using the Leech [Lee87] construction of inverse monoid of fractions. All the examples are classic.

1 Introduction

Abstract division categories were introduced by Leech [Lee87] as a small category C having finite pushouts, all of whose morphisms are epimorphisms, and with a quasi initial object I (Hom(I,A)$\neq\phi$ for any A$\in$ObC). The concept of the division category was originally introduced to help study semigroup coextensions (see [Lee75]). Mobius categories were introduced by Leroux [Ler75]. It was created a general program to extend the theory of Mobius functions. A Mobius category C is a decomposition finite category (a small category having finitely many decompositions $\alpha=\beta\cdot\gamma$ for any morphism α) such that an incidence function ξ of the incidence algebra of C has a convolution inverse iff $\xi(1_A)\neq 0$ for all A$\in$ObC. Mobius categories can also be characterized with inner conditions of a decomposition finite category (see [CLL80] and [Ler75]). We call a small category C an *abstract Mobius-division category* if C it is both a division category and a Mobius category. A (abstract) Mobius-division category (short, a MD category) C is a small category satisfying the following conditions (see [Sch05]):

(MD1) Every morphism of C is an epimorphism;
(MD2) C has pushouts;
(MD3) C has a quasi initia object I (i.e. Hom(I,A)≠ϕ for any object A∈ObC);
(MD4) The identity morpfisms are indecomposable ($1_A=\alpha\beta\Rightarrow\alpha=\beta=1_A$);
(MD5) C is decoposition-finite (i.e. the set $<\alpha>=\{(\beta,\gamma)\in MorC\times MorC| \alpha=\beta\gamma\}$ is finite for any morphism $\alpha\in MorC$.

Division categories (small categories satisfying (MD1),(MD2) and (MD3)) and inverse monoids are equivalent structures. Each one can be reconstructed from the other. Let S be an inverse monoid and C(S) the standard division category of S, and let D be a division category with quasi initial object I and L(D,I) the Leech inverse monoid of the pair (D,I). Then the above constructions (see [Lee87] or [Gri95, Chapter VII, Section 8])

$$S\longrightarrow C(S)\longrightarrow L(C(S),1) \quad and \quad (D,I)\longrightarrow L(D,I)\longrightarrow C(L(D,I))$$

leads to

$$S\cong L(C(S),1) \quad and \quad (D,I)\approx(C(L(D,I)),1),$$

where 1 denote the identity element of the corresponding monoid, ≅ denote a monoid isomorphism and ≈ denote a division category equivalence (i.e. an equivalence of categories which preserves the quasi initial objects).

A reduced standard division category $C_F(S)$ of an inverse monoid S relative to an idempotent transversal F of the *D*-classes of S with 1∈F is a full subcategory of C(S) and it is a division category (see [JL99]). If S is a combinatorial inverse monoid and the set of idempotents E(S) with the natural partial order ≤ on an inverse semigroup is locally finite then $C_F(S)$ is a Mobius category(see [Sch04a]).

So, the reduced standard division category $C_F(S)$ of a combinatorial inverse monoid with (E(S),≤) locally finite is a special Mobius category namely Mobius-division category. We have proved (see [Sch05],Theorem 3.3]), using the Leech constructions, that for a Mobius-division category (D,I) the division category equivalence (D,I)≈(C(L(D,I),1) is a category isomorphism. Now, in this note we give a large description of this result namely: up to isomorphism the only Mobius-division categories are the reduced standard division categories of combinatorial inverse monoids.

We refer the reader to Grillet [Gri95] for all standard definitions and results from inverse semigroup theory and division category theory.

Equivalent variations of Mobius categories and further information on these categories are given in [CLL80], [Ler75], [Ler80], [Ler82], [LS81], [Sch98], [Sch03], [Sch04a], [Sch04b] and [Sch05].

2 The main result

By [Sch04a, Theorem 3.3] if S is a combinatorial inverse monoid with the poset of idempotents (E(S),≤) locally finite then a reduced standard division category $C_F(S)$ relative to an idempotent transversal F with $1 \in F$, is a Mobius-division category. Up to isomorphism, the only Mobius-division categories are these categories (i.e. reduced standard division categories arising from combinatorial inverse monoids)

Theorem. *Every Mobius-division category* C *is isomorphic to a reduced standard division category* $C_F(S)$ *of a combinatorial inverse monoid* S *with the poset of idempotents* (E(S),≤) *locally finite.*

Proof. Let C be an abstract Mobius-division category with I a quasi initial object. Put

$$S = \{(\alpha,\beta) \in MorC \times MorC \mid Dom\,\alpha = Dom\,\beta = I; Codom\,\alpha = Codom\,\beta\}$$

with the multiplication defined by

$$(\alpha,\beta)\cdot(\alpha',\beta') = (p\alpha, q\beta'),$$

where

$$\begin{array}{ccc} I & \xrightarrow{\alpha'} & Codom\,\alpha' \\ {\scriptstyle\beta}\downarrow & & \downarrow{\scriptstyle q} \\ Codom\,\beta & \xrightarrow[p]{} & \bullet \end{array}$$

is a pushout.

The associativity of the multiplication follows by juxtaposing pushout diagrams. Since

$$\begin{array}{ccc} I & \xrightarrow{\alpha} & Codom\,\alpha \\ {\scriptstyle 1_I}\downarrow & & \downarrow{\scriptstyle 1_{Codom\,\alpha}} \\ I & \xrightarrow[\alpha]{} & Codom\,\alpha \end{array} \quad and \quad \begin{array}{ccc} I & \xrightarrow{1_I} & I \\ {\scriptstyle\beta}\downarrow & & \downarrow{\scriptstyle\beta} \\ Codom\,\beta & \xrightarrow[1_{Codom\,\beta}]{} & Codom\,\beta \end{array}$$

are pushouts, it follows that $(1_I,1_I)$ is the identity element of the semigroup S. Now, $(\alpha,\beta)\in S$ is an idempotent if and only if the diagram

$$\begin{array}{ccc} I & \xrightarrow{\alpha} & Codom\alpha = Codom\beta \\ {\scriptstyle\beta}\downarrow & & \downarrow{\scriptstyle 1_{Codom\alpha}} \\ Codom\alpha = Codom\beta & \xrightarrow[1_{Codom\beta}]{} & Codom\alpha = Codom\beta \end{array}$$

is a pushout, that is, $(\alpha,\beta)\in S$ is an idempotent if and only if $\alpha=\beta$.

To show that the idempotents commute, let (α,α), $(\beta,\beta)\in E(S)$. If the diagram

$$\begin{array}{ccc} I & \xrightarrow{\beta} & \cdot \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle q} \\ \cdot & \xrightarrow[p]{} & \cdot \end{array}$$

is a pushout, then

$$(\alpha,\alpha)\cdot(\beta,\beta)=(p\alpha,q\beta)=(q\beta,p\alpha)=(\beta,\beta)\cdot(\alpha,\alpha).$$

To show that S is an inverse monoid, let $(\alpha,\beta),(\beta,\gamma),(\alpha,\gamma)\in S$. Since

$$\begin{array}{ccc} I & \xrightarrow{\beta} & Codom\beta \\ {\scriptstyle\beta}\downarrow & & \downarrow{\scriptstyle 1_{Codom\beta}} \\ Codom\beta & \xrightarrow[1_{Codom\beta}]{} & Codom\beta \end{array}$$

is a pushout, it follows that $(\alpha,\beta)\cdot(\beta,\gamma)=(\alpha,\gamma)$. Thus

$$(\alpha,\beta)\cdot(\beta,\alpha)\cdot(\alpha,\beta)=(\alpha,\alpha)\cdot(\alpha,\beta)=(\alpha,\beta)$$

and

$$(\beta,\alpha)\cdot(\alpha,\beta)\cdot(\beta,\alpha)=(\beta,\beta)\cdot(\beta,\alpha)=(\beta,\alpha).$$

Hence S is a regular monoid and its idempotents commute. Thus, S is an inverse monoid and $(\alpha,\beta)^{-1}=(\beta,\alpha)$.

Now, (α_1,β_1), $(\alpha_2,\beta_2)\in S$ are *R*-related if and only if $(\alpha_1,\beta_1)\cdot(\alpha_1,\beta_1)^{-1}= (\alpha_2,\beta_2)\cdot(\alpha_2,\beta_2)^{-1}$ and they are *L*-related if and only if $(\alpha_1,\beta_1)^{-1}(\alpha_1,\beta_1) = (\alpha_2,\beta_2)^{-1}(\alpha_2,\beta_2)$. Thus,

$$(\alpha_1,\beta_1)R(\alpha_2,\beta_2) \Leftrightarrow \alpha_1 = \alpha_2 \quad \text{and} \quad (\alpha_1,\beta_1)L(\alpha_2,\beta_2) \Leftrightarrow \beta_1 = \beta_2.$$

It follows that the Green relation $H=R \cap L$ is the equality relation, and therefore S is a combinatorial inverse monoid. Also, it is straightforward to check that the natural partial order $\leq$ on the set of idempotents E(S) is given by

$$(\alpha,\alpha) \leq (\beta,\beta) \Leftrightarrow \alpha = q\beta$$

for some $q\in$MorC. Since C is decomposition-finite, it follows that (E(S),$\leq$) is locally finite.

We now prove that two idempotents $(\alpha,\alpha),(\beta,\beta)\in E(S)$ are *D*-related if and only if Codomα=Codomβ. If $(\alpha,\alpha)D(\beta,\beta)$ then there exists $(\alpha_1,\beta_1)\in S$ such that $(\alpha,\alpha)R(\alpha_1,\beta_1)$ and $(\alpha_1,\beta_1)L(\beta,\beta)$. Thus $\alpha=\alpha_1$ and $\beta=\beta_1$. But Codomα_1=Codomβ_1 and therefore Codomα=Codomβ. Conversely, if Codomα=Codomβ, then $(\alpha,\beta)\in S$ and $(\alpha,\alpha)R(\alpha,\beta)$, $(\alpha,\beta)L(\beta,\beta)$, that is $(\alpha,\alpha)D(\beta,\beta)$.

Let

$$F = \{(\alpha_A,\alpha_A) \mid Dom\alpha_A = I, Codom\alpha_A = A\}_{A\in ObC}$$

be an idempotent transversal of the *D*-classes of S with $(1_I,1_I)\in F$. By Theorem….., the reduced standard division category $C_F(S)$ is a Mobius-division category. The functor $G:C\rightarrow C_F(S)$ defined by

$$G(A) = (\alpha_A,\alpha_A)$$

and

$$G(\gamma) = ((\alpha_{Codom\gamma},\gamma\alpha_{Dom\gamma}),(\alpha_{Dom\gamma},\alpha_{Dom\gamma}))$$

is well-defined:

$$(\alpha_{Codom\gamma},\gamma\alpha_{Dom\gamma})^{-1}(\alpha_{Codom\gamma},\gamma\alpha_{Dom\gamma}) = (\gamma\alpha_{Dom\gamma},\gamma\alpha_{Dom\gamma}) \leq (\alpha_{Dom\gamma},\alpha_{Dom\gamma})$$

and

$$(\alpha_{Codom\gamma},\gamma\alpha_{Dom\gamma})\cdot(\alpha_{Codom\gamma},\gamma\alpha_{Dom\gamma})^{-1} = (\alpha_{Codom\gamma},\alpha_{Codom\gamma}),$$

that is

$$(\alpha_{Codom\gamma}, \gamma\alpha_{Dom\gamma}) \in Hom(G(Dom\gamma), G(Codom\gamma)).$$

Since

$$\begin{array}{ccc} I & \xrightarrow{\alpha_{Codom\beta}} & Codom\beta \\ \downarrow{\scriptstyle \gamma\alpha_{Dom\gamma}} & & \downarrow{\scriptstyle \gamma} \\ Codom\gamma & \xrightarrow[1_{Codom\gamma}]{} & Codom\gamma \end{array}$$

is a pushout if the composition γβ make sense, it follows

$$G(\gamma)G(\beta) =$$
$$= ((\alpha_{Codom\gamma}, \gamma\alpha_{Dom\gamma}), (\alpha_{Dom\gamma}, \alpha_{Dom\gamma}))((\alpha_{Codom\beta}, \beta\alpha_{Dom\beta}), (\alpha_{Dom\beta}, \alpha_{Dom\beta})) =$$
$$= ((\alpha_{Codom\gamma}, \gamma\alpha_{Dom\gamma}) \cdot (\alpha_{Codom\beta}, \beta\alpha_{Dom\beta}), (\alpha_{Dom\beta}, \alpha_{Dom\beta})) =$$
$$= ((\alpha_{Codom\gamma}, \gamma\beta\alpha_{Dom\beta}), (\alpha_{Dom\beta}, \alpha_{Dom\beta})) =$$
$$= ((\alpha_{Codom\gamma\beta}, \gamma\beta\alpha_{Dom\gamma\beta}), (\alpha_{Dom\gamma\beta}, \alpha_{Dom\gamma\beta})) = G(\gamma\beta),$$

and obviously

$$G(1_A) = ((\alpha_A, \alpha_A), (\alpha_A, \alpha_A)) = 1_{G(A)}.$$

So, G is a well-defined covariant functor. It is straightforward to check that G is an equivalence which produces a bijection between the objects of C and $C_F(S)$. Hence C and $C_F(S)$ are isomorphic categories as required. ■

A monoid is a category with a single object. The monoid $(\mathbf{N}^*, \cdot)$ is a Mobius-division category with a single object. The diagram

$$\begin{array}{ccc} \bullet & \xrightarrow{m} & \bullet \\ {\scriptstyle n}\downarrow & & \downarrow{\scriptstyle \frac{[m,n]}{m}} \\ \bullet & \xrightarrow[\frac{[m,n]}{n}]{} & \bullet \end{array}$$

is a pushout (where [m,n]=l.c.m.(m,n)). The corresponding combinatorial inverse monoid S from the previous Theorem is the following one:

$$S = N^* \times N^* \qquad (m,n)\cdot(m',n') = \left(\frac{[m',n]m}{n}, \frac{[m',n]n'}{m'}\right)$$

This is a bisimple (with a single *D*-classes) combinatorial inverse monoid called the multiplicative analogue of the bicyclic semigroup. The reduced standard division category of this semigroup is the Mobius-division monoid (**N***,·). The incidence algebra of this Mobius category is the Dirichlet algebra of the arithmetical functions. So, the Mobius function of the multiplicative analogue of the bicyclic semigroup is just the classical Mobius function.

The monoid (**N**,+) is also a Mobius-division category. The diagram

$$\begin{array}{ccc} \bullet & \xrightarrow{m} & \bullet \\ {\scriptstyle n}\downarrow & & \downarrow \;{\scriptstyle \max(m,n)-m} \\ \bullet & \xrightarrow[\max(m,n)-n]{} & \bullet \end{array}$$

is a pushout. The corresponding combinatorial inverse monoid S is given by:

$$S = N \times N \qquad (m,n)\cdot(m',n') = (\max(m',n) - n + m, \max(m',n) - m' + n')$$

This is the bicyclic semigroup and its reduced standard division category is the Mobius monoid (**N**,+). The incidence algebra of this Mobius category is the Cauchy algebra of arithmetical functions. Thus the Mobius function of the bicyclic semigroup is the following one

$$\mu(n) = \begin{cases} 1 & if \quad n = 0 \\ -1 & if \quad n = 1 \\ 0 & if \quad n \geq 2 \end{cases}$$

Now, let <u>C</u> be the category given by

- Ob<u>C</u>=**N** ;
- Hom(m,n)={(a,b,m)|a,b∈**N**, a+b+m=n}
- m —(a,b,m)→n—(a',b',n)→p = m—(a'+a, b'+b, m)→p.

The category <u>C</u> is a Mobius-division category with 0 a quasi initial object. We denote a morphism (a,b,0):0→a+b by (a,b). Then the inner square

$$\begin{array}{ccc}
0 & \overset{(c_2,d_2)}{\underset{(a_2,b_2)}{\rightrightarrows}} & n_2 = a_2 + b_2 = c_2 + d_2 \\
{\scriptstyle (a_1,b_1)}\downarrow \ \downarrow{\scriptstyle (c_1,d_1)} & & \downarrow {\scriptstyle (\max(c_1,a_2)-a_2,\max(d_1,b_2)-b_2,n_2)} \\
a_1 + b_1 = c_1 + d_1 = n_1 & \xrightarrow[\max(c_1,a_2)-c_1,\max(d_1,b_2)-d_1,n_1]{} & \bullet
\end{array}$$

is a pushout. The corresponding combinatorial inverse monoid is the following

$$S = \{(a,b,c,d) \in N^4 \mid a + b = c + d\}$$

with the multiplication

$$(a_1,b_1,c_1,d_1)\cdot(a_2,b_2,c_2,d_2) =$$
$$(\max(c_1,a_2) - c_1 + a_1, \max(d_1,b_2) - d_1 + b_1, \max(c_1,a_2) - a_2 + c_2, \max(d_1,b_2) - b_2 + d_2).$$

This is the free monogenic inverse semigroup and its reduced standard division category is the Mobius-division category $\underline{C}$. The Mobius function and the Mobius inversion formula of the free monogenic inverse monoid are given in [Sch04b].

References

[AS99] **E.R.Aznar, A.M.Sevilla**, *Equivalences between D-categories of inverse monoids,* Semigroup Forum 59 (1999), 435-452.

[CLL80] **M.Content, F.Lemay, P.Leroux**, *Categories de Mobius et fonctorialites: un cadre general pour l'inversion de Mobius,* J.Comb.Th.(A) 28 (1980), 169-190.

[Gri95] **P.A.Grillet**, *Semigroups An introduction to the structue theory,* Marcel Dekker Inc., New York, 1995.

[JL99] **H. James, M.V.Lawson**, *An application of grupoids of fractions to inverse semigroups,* Per. Math. Hungarica, 38 (1999), 43-54.

[Lee75] **J.Leech**, *H-coextensions of monoids,* Memoirs AMS, 157 (1975), 1-66.

[Lee87] **J.Leech**, *Constructing inverse monoids from small categories,* Semigroup Forum 36 (1987), 89-116.

[Lee98] **J. Leech**, *On the foundations of inverse monoids and inverse algebras,* Proc. Edinburgh Math.soc. 41 (2) (1998), 1-21.

[Ler75] **P. Leroux**, *Les categories de Mobius*, Cahiers de Topologie et Geom. Differentielle 16 (1975), 280-282.

[Ler80] **P. Leroux**, *Categories triangulaires: exemples, applications et problemes* Rapport de recherché, Univ.du Quebec a Montreal, 1980.

[Ler82] **P. Leroux**, *The isomorphism problem for incidence algebras of Mobius categories,* Illinois J. of Math. 26 (1982), 52-61.

[LS81] **P. Leroux, J.Sarraille**, *Structure of incidence algebras of graphs,* Comm. Algebra 19 (1981), 1479-1517.

[Log81] **M. Loganathan**, *Cohomology of inverse semigroups,* J.Algebra 70 (1981), 375-393.

[Sch98] **E.D. Schwab**, *Complete multiplicativity and complete additivity in Mobius categories,* Italian J. of Pure and Appl. Math. 3 (1998), 37-48.

[Sch03] **E.D. Schwab**, *On triangular categories,* Houston J. Math. 29 (2003), 25-40.

[Sch04a] **E.D. Schwab**, *Mobius categories as reduced standard division categories of combinatorial inverse monoids*, Semigroup Forum, 69 (2004), 30-40.

[Sch04b] **E.D. Schwab**, *The Mobius category of some combinatorial inverse semigroups,* Semigroup Forum, 69 (2004), 41-50.

[Sch05] **E.D. Schwab**, *On incidence algebras of combinatorial inverse monoids,* (to appear).