

A Partial Order on Bipartite Graphs with n Vertices

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ABSTRACT. The paper examines a partial order on bipartite graphs (X_1, X_2, E) with n vertices, $X_1 \cup X_2 = \{1, 2, \dots, n\}$. The basis of such bipartite graph is $X_1 = \{1, 2, \dots, k\}$, $0 \leq k \leq n$. If $U = (X_1, X_2, E(U))$ and $V = (Y_1, Y_2, E(V))$ then $U \leq V$ iff $|X_1| \leq |Y_1|$ and $\{(i, j) \in E(U) : j > |Y_1|\} = \{(i, j) \in E(V) : i \leq |X_1|\}$. This partial order is a natural partial order of subobjects of an object in a triangular category with bipartite graphs as morphisms.

KEYWORDS: Bipartite graph; partial order; triangular category.

1 The set B_n of bipartite graphs

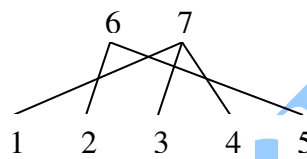
We restrict attention to finite simple graph and use standard notations and definitions of graph theory. A graph G is a pair (X, E) , where X is a set $\{x_1, x_2, \dots, x_n\}$ of elements called vertices, and E is a set of pairs of vertices $(x_i, x_j) = (x_j, x_i)$. An element (x_i, x_j) of E is called an edge of $G = (X, E)$. Any two vertices, x_i and x_j , are said to be adjacent if and only if the pair (x_i, x_j) is an edge of G . A graph (X, E) is bipartite if its vertices can be partitioned into two sets X_1 and X_2 ($X_1 \cup X_2 = X$; $X_1 \cap X_2 = \emptyset$) such that no two vertices in the same set are adjacent. One often writes $U = (X_1, X_2, E(U))$ to denote a bipartite graph, and we say that the first set X_1 is the basis of the bipartite graph U .

An isomorphism of graphs $G = (X, E)$ and $G' = (X', E')$ is a bijection $f : X \rightarrow X'$ such that any two vertices $x_i, x_j \in X$ are adjacent in G if and only if $f(x_i), f(x_j) \in X'$ are adjacent in G' .

Now, we denote by B_n the set of bipartite graphs $U=(X_1, X_2, E(U))$ with n vertices such that the following laws hold:

- 1) The family $\{x_1, x_2, \dots, x_n\}$ of the vertices of U is denoted by its set of indices $\{1, 2, \dots, n\}$ such that the first indices $(i = 1, 2, \dots, k)$ are in the same partite set namely in the basis X_1 of U and $X_2 = \{k + 1, k + 2, \dots, n\}$.
- 2) If i and j are adjacent in U such that $i \in X_1$ and $j \in X_2$ then (i, j) denotes the corresponding edge of U . Thus $(i, j) \in E(U)$ implies $i < j$.

For instance, the following bipartite graph U :



with the set of edges $E = \{(1,7), (2,6), (3,7), (4,6), (5,6)\}$ and $X_1 = \{1, 2, 3, 4, 5\}$, $X_2 = \{6, 7\}$ is an element of B_7 . The following two elements of B_7 :



are two distinct elements of B_7 . The first has basis the empty set, and $\{1, 2, 3, 4, 5, 6, 7\}$ is the basis of the second bipartite graph.

It is straightforward to check that

$$|B_n| = \sum_{k=0}^n 2^{k(n-k)},$$

where $|B_n|$ is the cardinality of B_n .

2. A partial order on B_n

Let $U = (X_1, X_2, E(U))$ and $V = (Y_1, Y_2, E(V))$ be two elements of B_n .

Proposition 1. *The relation defined by*

$U \leq V \Leftrightarrow |X_1| \leq |Y_1|$ and $\{(i, j) \in E(U) \mid j > |Y_1|\} = \{(i, j) \in E(V) \mid i \leq |X_1|\}$
is a partial order on B_n .

Proof. It is immediate that the relation defined above is reflexive and anti-symmetric. To show that it is also transitive, let $U = (X_1, X_2, E(U))$, $V = (Y_1, Y_2, E(V))$, $W = (Z_1, Z_2, E(W)) \in B_n$ such that $U \leq V$ and $V \leq W$. It follows that

a)

$$|X_1| \leq |Y_1| \leq |Z_1|,$$

b)

$$\begin{aligned} & (i_0, j_0) \in \{(i, j) \in E(U) \mid j > |Z_1|\} \Rightarrow \\ \Rightarrow & (i_0, j_0) \in \{(i, j) \in \{(i, j) \in E(U) \mid j > |Y_1|\} = \{(i, j) \in E(V) \mid i \leq |X_1|\} \\ \Rightarrow & (i_0, j_0) \in \{(i, j) \in E(V) \mid i \leq |X_1| \text{ and } j > |Z_1|\} = \{(i, j) \in E(W) \mid i \leq |X_1|\} \\ \Rightarrow & \{(i, j) \in E(U) \mid j > |Z_1|\} \subseteq \{(i, j) \in E(W) \mid i \leq |X_1|\} \end{aligned}$$

c)

$$\begin{aligned} & (i_0, j_0) \in \{(i, j) \in E(W) \mid i \leq |X_1|\} \Rightarrow \\ \Rightarrow & (i_0, j_0) \in \{(i, j) \in E(W) \mid i \leq |Y_1|\} = \{(i, j) \in E(V) \mid j > |Z_1|\} \\ \Rightarrow & (i_0, j_0) \in \{(i, j) \in E(V) \mid i \leq |X_1| \text{ and } j > |Z_1|\} = \{(i, j) \in E(U) \mid j > |Z_1|\} \\ \Rightarrow & \{(i, j) \in E(W) \mid i \leq |X_1|\} \subseteq \{(i, j) \in E(U) \mid j > |Z_1|\} \end{aligned}$$

a), b) and c) implies that $U \leq W$.

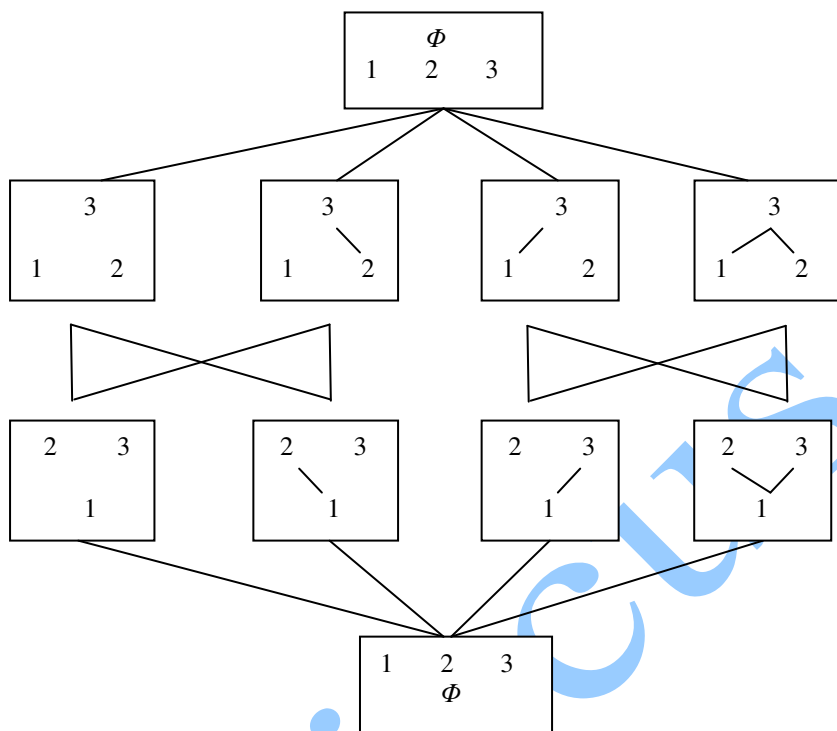
Proposition 2. *If the bases of two elements $U, V \in B_n$, $U \neq V$, are equal then U and V are incomparable.*

Proof. The equality

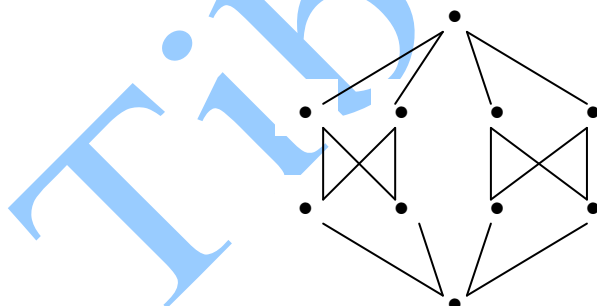
$$\{(i, j) \in E(U) \mid j > |Y_1|\} = \{(i, j) \in E(V) \mid i \leq |X_1|\}$$

where $X_1 = Y_1$, implies that $U=V$.

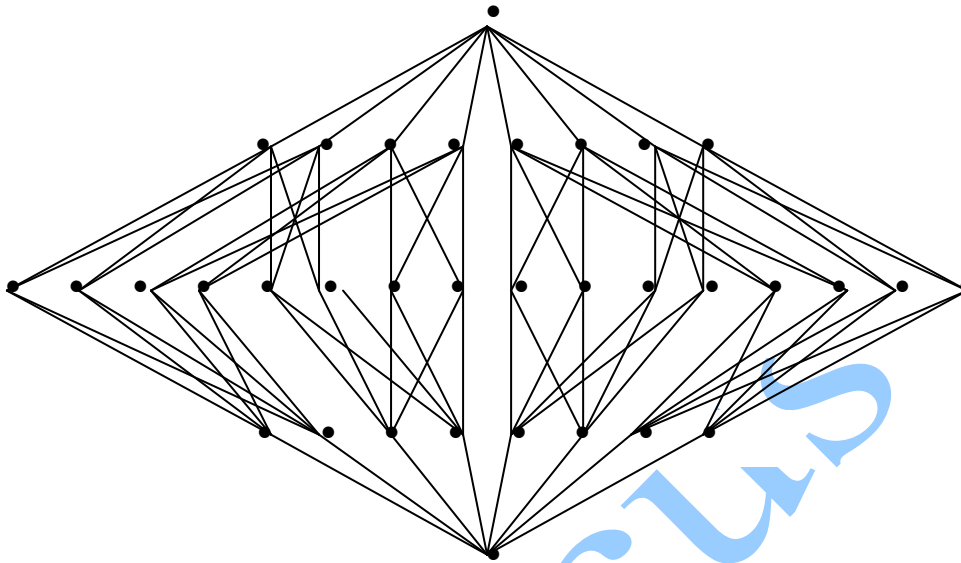
Now, let $n=3$. Then the Hasse diagram of the partially ordered set (B_3, \leq) is the following one:



Without specifying the bipartite graphs, the Hasse diagram of (B_3, \leq) is given by:



The case $n > 3$ is somewhat laborious. For example, the Hasse diagram of the partial ordered set (B_4, \leq) is the following one:



3. Connection with a triangular category

Möbius inversion for categories was considered for the first time by Leroux [Ler75]. A Möbius category in the sense of Leroux is a decomposition finite category C (i.e. a small category where each morphism α has only finitely many nontrivial factorizations) such that an incidence function $f : MorC \rightarrow$ has a convolution inverse if and only if $f(1_A) \neq 0$ for any identity morphism 1_A of C .

The convolution $f * g$ of two incidence functions f and g is defined by

$$(f * g)(\alpha) = \sum_{\alpha = \beta\gamma} f(\beta) \cdot g(\gamma) \quad (\alpha \in MorC).$$

Möbius categories have also been characterized as decomposition-finite categories in which

- (1) each identity morphism is indecomposable, i.e., $1_A = \beta\gamma$ implies $\beta = 1_A = \gamma$;
- (2) $\beta\gamma = \gamma$ implies that β is identity morphism.

Now, it is straightforward to see that a special class of categories (called triangular categories by Leroux [Ler80]) in which the set of objects is the set of nonnegative integers \mathbb{N} and the family of numbers $|Hom(k,n)|$

(where $|Hom(k,n)|$ is the number of morphisms from k to n) constitute a triangular family of numbers, that is:

$$|Hom(n,n)| = 1 \text{ for all } n \in N; \text{ and } |Hom(m,n)| = 0 \text{ if } m > n.$$

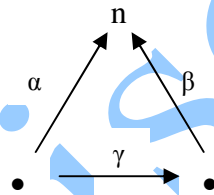
The prime example of a triangular category (denoted Δ in [Ler80]) is that for which $0 \in N$ is the initial object and $Hom_{\Delta}(k,n)$ is "the set of all injective and isotone maps from $\{1,2,\dots,k\}$ to $\{1,2,\dots,n\}$ ". The corresponding triangular family of numbers is the following one:

$$|Hom_{\Delta}(k,n)| = \binom{n}{k} \quad (k \leq n)$$

More combinatorial triangular families of numbers can be represented by triangular categories (see [Ler80], [Ler90]).

Let C be a triangular category. The set $S(n)$ of subobjects of $n \in N$ (or, rather, monomorphisms into n) is an ordered set:

$$\alpha \leq \beta \Leftrightarrow \exists \gamma : \beta \gamma = \alpha$$



This relation is called the natural partial order on the set of subobjects of n .

Proposition 3. *Since $|Hom(k,k)| = 1$ for every $k \in N$, two monomorphisms α, β into n , $\alpha \neq \beta$, with the same domain k are incomparable.*

A triangular category is called lattice-triangular if $(S(n), \leq)$ is a lattice for every $n \in N$. A triangular category C is called monomorphic-triangular if any morphism of C is a monomorphism. We have:

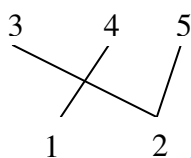
Theorem 4. ([Sch03]) *Let C be a monomorphic-triangular category. Then C is lattice-triangular if and only if C has pullbacks.*

The triangular category Δ is a lattice triangular category. It is straightforward to check that in Δ the lattice $(S(n), \leq)$ is isomorphic to the Boolean algebra of all subsets of the set $\{1,2,\dots,n\}$. This category is not a category with pushout and therefore in Theorem 1, the "pullback" cannot be replaced by "pushout".

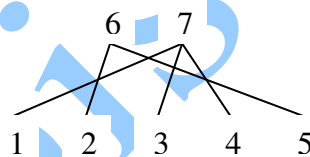
Now, we shall consider the category B of bipartite graphs (see *Bipis* in [Ler80]) defined by:

- $ObB=N$;
- $Hom_B(k,n) = \begin{cases} \{U \in B_n \mid \{1,2,\dots,k\} \text{ is the basis of } U\} & \text{if } k \leq n \\ \emptyset & \text{if } k > n \end{cases}$
- The composition of two morphisms: if $U \in Hom_B(m,k)$ and $V \in Hom_B(k,n)$ then the composition $V \bullet U$ is the bipartite graph with n vertices, $1,2,\dots,n$, having the basis $\{1,2,\dots,m\}$; and the set of edges being the union of the set of edges of U and the set of those edges of V which have an endpoint in the basis of U .

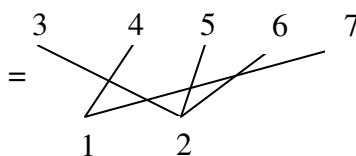
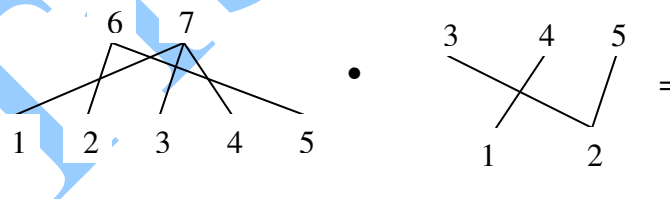
For example, if $U \in Hom_B(2,5)$ is given by



and if $V \in Hom_B(5,7)$ is given by



then



that is,

$$E(V \bullet U) = \{(1,4); (2,3); (2,5); (1,7); (2,6)\}.$$

Hence,

$$E(V \bullet U) = E(U) \cup \{(i, j) \in E(V) \mid i \leq m\}.$$

The identity morphism from n to n is the bipartite graph with n vertices and with basis $\{1, 2, \dots, n\}$; the set of edges being the empty set.

Proposition 5. *The category B is a triangular category and the corresponding triangular family of numbers is the following one:*

$$\mid \text{Hom}_B(k, n) \mid = 2^{k(n-k)} \quad (k \leq n)$$

Proposition 6. *The category B is monomorphic but it is not epimorphic.*

Proof. Let $V \in \text{Hom}_B(m, n)$ and $U, U' \in \text{Hom}_B(k, m)$ be such that

$$V \bullet U = V \bullet U'.$$

Then,

$$U \cup \{(i, j) \in V \mid i \leq k\} = U' \cup \{(i, j) \in V \mid i \leq k\}$$

and therefore $U = U'$.

Now, if $V' \in \text{Hom}_B(m, n)$ such that

$$V \bullet U = V' \bullet U$$

then,

$$\{(i, j) \in V \mid i \leq k\} = \{(i, j) \in V' \mid i \leq k\}.$$

But this does not imply that $V = V'$.

Proposition 7. *The partial order on B_n is the natural partial order on the set of subobjects of n in the triangular category B .*

Proof. Let $U = (X_1, X_2, E(U))$ and $V = (Y_1, Y_2, E(V))$ be two elements of B_n such that

$$\mid X_1 \mid \leq \mid Y_1 \mid \text{ and } \{(i, j) \in E(U) \mid j > \mid Y_1 \mid\} = \{(i, j) \in E(V) \mid i \leq \mid X_1 \mid\}.$$

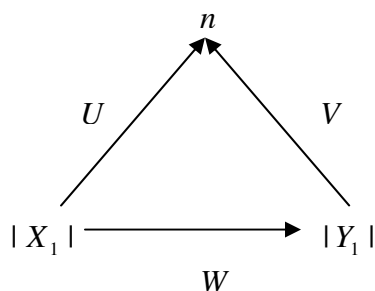
These two bipartite graphs U and V are two morphisms of the category B having the same codomain n . Consider the morphism $W = (X_1, Y_1 - X_1, E(W)) \in \text{Hom}_B(\mid X_1 \mid, \mid Y_1 \mid)$, where

$$E(W) = \{(i, j) \in E(U) : j \leq \mid Y_1 \mid\}$$

and we obtain:

$$\begin{aligned} E(V \bullet W) &= E(W) \cup \{(i, j) \in E(V) : i \leq \mid X_1 \mid\} = \\ &= \{(i, j) \in E(U) : j \leq \mid Y_1 \mid\} \cup \{(i, j) \in E(U) : j > \mid Y_1 \mid\} = E(U). \end{aligned}$$

It follows that the diagram



is commutative in B .

Conversely, if the above diagram is commutative in B , where $U = (X_1, X_2, E(U))$, $V = (Y_1, Y_2, E(V))$ and $W = (X_1, Y_1 - X_1, E(W))$, then

$$|X_1| \leq |Y_1|$$

and

$$\left. \begin{aligned} E(W) &= \{(i, j) \in E(U) : j \leq |Y_1|\} \\ E(U) &= E(V \bullet W) = E(W) \cup \{(i, j) \in E(V) : i \leq |X_1|\} \end{aligned} \right\} \Rightarrow$$

$$\Rightarrow E(U) = \{(i, j) \in E(U) : j \leq |Y_1|\} \cup \{(i, j) \in E(V) : i \leq |X_1|\}$$

$$\Rightarrow \{(i, j) \in E(U) : j > |Y_1|\} = \{(i, j) \in E(V) : i \leq |X_1|\}$$

as required.

Taking into account the Hasse diagram of (B_3, \leq) and Proposition 7, it follows:

Proposition 8. *The monomorph-triangular category B is not a lattice-triangular category.*

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